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GEOMETRY

AND

COLLINEATION GROUPS

OF THE

FINITE PROJECTIVE PLANE

PG$(2,2^2)$

A DISSERTATION
Presented to the Faculty of Princeton University
in Candidacy for the Degree of
Doctor of Philosophy

(DEPARTMENT OF MATHEMATICS)

BY

ULYSSES GRANT MITCHELL

PRINCETON
1910

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ERRATA

Page 6, line 21, add $+a_{22}a_{13}^2$ to the left member of the equation.
Page 11, line 4, in place of "$PG(2,2^n)$" read "$PG(2,2\ )$."
Page 12, line 10, in place of "$\rho x_3'=x_2+x_3$" read "$\rho x_3'=ix_2+x_3$."
Page 12, line 11, in place of "$(\rho^2+\rho+1)$" read "$(\rho^2+\rho+i)$."
Page 18, line 30, in place of "*on*" read "*or*."
Page 20, line 23, in place of "I_0" read "I_3" (twice).

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GEOMETRY AND COLLINEATION GROUPS OF THE FINITE PROJECTIVE PLANE PG $(2,2^2)$.*

§1. Definition of a Finite Projective Plane.

The definition and general properties of finite projective spaces together with references to the literature of the subject may be found in a paper by VEBLEN and BUSSEY in the *Transactions of the American Mathematical Society,* Vol. 7, pp. 241-259. They used the symbol PG(k,p^n), where k,p,n are integers and p is a prime, to indicate a finite projective space of k dimensions having p^n+1 points to the line. It is the purpose of this paper to discuss some of the properties of the PG$(2,2^n)$ and to determine all subgroups of the group of projective collineations in PG$(2,2^2)$.

We give a brief summary of the analytic and synthetic definitions of a finite projective plane.

If x_1,x_2,x_3, are marks of a Galois field† [designated by GF(p^n)] of order p^n there are $(p^{3n}-1)/(p-1)=p^{2n}+p^n+1$ elements of the form (x_1,x_2,x_3) provided that the elements (x_1,x_2,x_3) and (lx_1,lx_2,lx_3) indicate the same element

* Presented to the American Mathematical Society, April 29, 1911

† For definition and properties of a Galois field see E. H. MOORE, SUBGROUPS OF THE GENERALIZED FINITE MODULAR GROUP, University of Chicago Dec. Pub, Vol. IX., pp. 141-156; L. E. DICKSON, LINEAR GROUPS, pp. 1-14.

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when l is any mark other than zero and provided that $(0,0,0)$ be excluded from consideration. These elements constitute a finite projective plane if the equation

$$u_1x_1+u_2x_2+u_3x_3=0$$

[the domain for coefficients and variables being the GF(p^n)] be taken as the equation of a line except when $u_1=u_2=u_3=0$. The line is denoted by the symbol (u_1,u_2,u_3) and the symbols (u_1,u_2,u_3) and (lu_1,lu_2,lu_3) where l is any mark other than zero denote the same line. The points of a line are those points whose coordinates (x_1,x_2,x_3) satisfy its equation.

Taking $0, 1$, i and i^2 for the marks of the GF (2^2) where i is defined as a root of the equation $i^2=i+1$ and hence $i^3\equiv 1$ (mod. 2) the PG$(2,2^2)$ so defined may be exhibited in the table of alignment given on the opposite page. In the analytical processes of PG$(2,2^n)$ no distinction need be made between plus and minus signs since $-1\equiv 1$ (mod. 2).

Synthetically a finite projective plane may be defined as a set of elements which for suggestiveness are called points, arranged in subsets called lines and subject to the following conditions:

I. The set contains a finite number, greater than one, of lines, and each line contains p^n+1 points (p and n integers and p a prime).

II. If A and B are distinct points there is one and only one line that contains A and B.

III. All the points considered are in the same plane.

From this definition it follows* that the principle of duality is valid in the plane so defined, that there are p^n+1 lines through each point and that the total number of points in the plane is $p^{2n}+p^n+1$.

In a PG $(2,2^2)$ there are then 21 points and 21 lines. The following set of elements arranged in 21 lines of 5 elements each will be seen to satisfy the given synthetic definition and to be identical with the table given opposite.

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	0
4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	0	1	2	3
14	15	16	17	18	19	20	0	1	2	3	4	5	6	7	8	9	10	11	12	13
16	17	18	19	20	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15

§2. *Preliminary Theorems.*

Theorem 1. *In PG$(2,2^n)$ the diagonal points of a complete quadrangle are collinear.*

Proof. Let three of the vertices, A, B and C of the quadrangle (Fig. 1) be taken as the triangle of reference and let the fourth vertex, D, be assigned the coordinates $(1,1,1)$. The intersections of AB with CD, AC with BD and AD with

* Cf. VEBLEN and YOUNG, Projective Geometry, Vol. I, pp. 16-17.

Table of Alignment

		0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
0	$x_1=0$	0,1,i										0,1,1		0,0,1					0,1,0	0,i,1		
1	$ix_1+x_2+ix_3=0$		1,1,i										1,0,1		i,1,1					0,i,1	1,i,0	
2	$ix_1+x_2=0$			1,1,i										0,0,1		$1,i,i^2$					1,i,0	1,i,1
3	$ix_1+ix_2+x_3=0$	0,1,i			1,1,0										i,1,1		1,0,i					1,i,1
4	$ix_2+x_3=0$	0,1,i	1,1,i			1,0,0										$1,i,i^2$		i,1,i				
5	$ix_1+x_3=0$		1,1,i	1,i,i			$1,i^2,i$										1,0,i		0,1,0			
6	$x_1+x_2+ix_3=0$			1,i,i	1,1,0			i,0,1										i,1,i		0,i,1		
7	$x_3=0$				1,1,0	1,0,0			i,1,0										0,1,0		1,i,0	
8	$x_2+ix_3=0$					1,0,0	$1,i^2,i$			i,i,1										0,i,1		1,i,1
9	$x_1+i^2x_2+ix_3=0$	0,1,i					$1,i^2,i$	i,0,1			1,1,1										1,i,0	
10	$x_1+ix_2+ix_3=0$		1,1,i					i,0,1	i,1,0			0,1,1										1,i,1
11	$x_1+ix_2+x_3=0$	0,1,i		1,i,i					i,1,0	i,i,1			1,0,1									
12	$x_1+x_2=0$		1,1,i		1,1,0					i,i,1	1,1,1			0,0,1								
13	$x_2+x_3=0$			1,i,i		1,0,0					1,1,1	0,1,1			i,1,1							
14	$x_1+x_2+x_3=0$				1,1,0		$1,i^2,i$					0,1,1	1,0,1			$1,i,i^2$						
15	$x_2=0$					1,0,0		i,0,1					1,0,1	0,0,1			1,0,i					
16	$x_1+ix_2=0$						$1,i^2,i$		i,1,0					0,0,1	i,1,1			i,1,i				
17	$x_1+ix_3=c$							i,0,1		i,i,1					i,1,1	$1,i,i^2$			0,1,0			
18	$x_1+ix_2+i^2x_3=0$								i,1,0		1,1,1					$1,i,i^2$	1,0,i			0,i,1		
19	$ix_1+x_2+x_3=0$									i,i,1		0,1,1					1,0,i	i,1,i			1,i,0	
20	$x_1+x_3=0$										1,1,1		1,0,1					i,1,i	0,1,0			1,i,1

BC determine the diagonal points P,Q,R as $(1,0,1)$, $(1,1,0)$ and $(0,1,1)$ respectively, which are collinear on the line $x_1+x_2+x_3=0$.

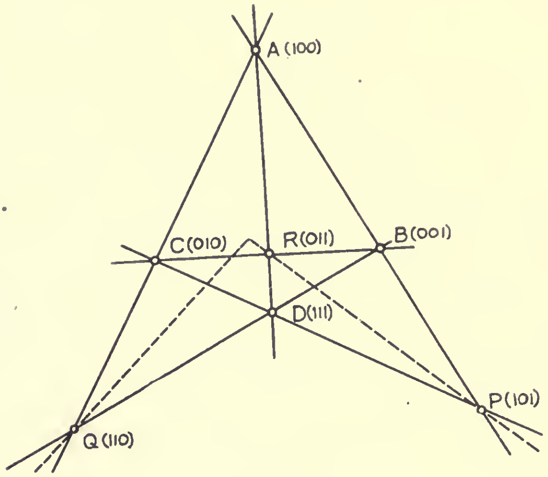

FIG. 1.

Since the quadrangle A,B,C,D is projectively equivalent to any other quadrangle in the plane the proof is complete.

The line joining the diagonal points of any complete quadrangle will be referred to as the *diagonal line* of the quadrangle.

Definition of conic. A *point conic* is defined as the locus of the points of intersection of corresponding lines in two projective non-perspective pencils of lines. A *line conic* is defined as consisting of the lines that join corresponding points in two projective non-perspective ranges of points. In PG $(2,2^n)$ the number of points in a point conic is 2^n+1, the number of lines in a pencil, and the number of lines in a line conic is 2^n+1, the number of points in a range. Hence in PG$(2,2^2)$ a point conic consists of any five points no three of which are collinear and a line conic consists of any five lines no three of which are concurrent.

A *tangent* to a point conic is defined as a line which has one and only one point in common with the conic. There is one and only one tangent at a given point on a conic since there are 2^n+1 lines through the point and 2^n lines joining it to other points of the conic.

By taking the equations of two projective pencils of lines $\lambda P+\mu Q=0$ and $\lambda P'+\mu Q'=0$ [$P=0$, $Q=0$, $P'=0$, $Q'=0$ being equations in abbreviated notation of lines in PG$(2,2^n)$ and λ and μ marks of the GF(2^n)] and eliminating λ and μ it is readily shown that the equation of a point conic is a homogeneous equation of the second degree in three variables with coefficients in the GF(2^n). Similarly, using line coordinates it may be shown that the equation of a line conic is also a homogeneous equation of the second degree in three variables with coefficients in the GF(2^n).

THEOREM 2. *Every equation of the form*

$$F(x_1,x_2,x_3)= \sum_{i,j}^{1,2,3} a_{ij}x_ix_j=0 \qquad (i\leq j)$$

where the coefficients are marks of the GF(2^n) is satisfied by the coordinates of at least one point in PG($2,2^n$).

Proof. Suppose neither a_{11} nor a_{12} to be zero. Taking $x_3=1$ the equation reduces to $a_{11}x_1^2+(a_{12}x_2+a_{13})x_1+a_{22}x_2^2+a_{23}x_2+a_{33}=0$ which is satisfied by

$$x_1=\frac{1}{a_{11}a_{12}}\sqrt{a_{11}(a_{13}^2a_{22}+a_{12}a_{13}a_{23}+a_{12}^2a_{33})}$$

and $x_2=\dfrac{a_{13}}{a_{12}}$ Moreover, since in the GF(2^n) every mark satisfies the equation $x^{2^n}\equiv x$ it is a perfect square and since $-1\equiv 1$ (mod. 2) its square root is unique. These values for x_1 and x_2 are therefore uniquely determined and lie in the GF(2^n). If $a_{11}=0$, $F(x_1,x_2,x_3)=0$ is satisfied by $(1,0,0)$ and if $a_{11}\neq 0$ and $a_{12}=0$ $F(x_1,x_2,x_3)=0$ is satisfied by $\left(\sqrt{\dfrac{a_{22}}{a_{11}^2}},1,0\right)$.

THEOREM 3. *Every equation of the form*

$$F(x_1,x_2,x_3)= \sum_{i,j}^{1,2,3} a_{ij}x_ix_j=0 \qquad (i\leq j)$$

where x_1,x_2,x_3 are point coordinates and the coefficients are marks of the GF(2^n) represents a point conic in PG($2,2^n$).

Proof. Since by the previous theorem $F(x_1x_2,x_3)=0$ is satisfied by at least one point in PG($2,2^n$), by means of a linear transformation of that point into the point $(0,0,1)$ $F(x_1,x_2,x_3)=0$ can be transformed into the equation

$$F_1(x_1,x_2,x_3) = b_{11}x_1^2+b_{12}x_1x_2+b_{22}x_2^2+b_{13}x_1x_3+b_{23}x_2x_3=0$$

This equation may be written

$$x_1(b_{11}x_1+b_{12}x_2+b_{13}x_3) + x_2(b_{22}x_2+b_{23}x_3)=0$$

which is seen to be the locus of points of intersection of corresponding lines in the two projective pencils of lines

$$lx_2+m(b_{11}x_1+b_{12}x_2+b_{13}x_3)=0 \text{ and } lx_1+m(b_{22}x_2+b_{23}x_3)=0$$

Hence $F(x_1,x_2,x_3)=0$ represents the locus previously defined as a point conic.

THEOREM 4. *In PG($2,2^n$) all tangents to the point conic*

$$F(x_1,x_2,x_3)\equiv \sum_{i,j}^{1,2,3} a_{ij}x_ix_j=0 \qquad (i\leq j)$$

are concurrent and their point of concurrence is (a_{23},a_{13},a_{12}).

Proof. The line $c_1x_1+c_2x_2+c_3x_3=0$, the domain for variables and coefficients being the GF(2^n), will be a tangent to the conic $F(x_1,x_2,x_3)=0$ in case it has but one point in common with the conic. Eliminating x_1 from the two equations gives

$$(a_{11}c_2^2 + a_{12}c_1c_2)\ x_2^2 + c_1\ (a_{12}c_3 + a_{13}c_2 + a_{23}c_1)\ x_2x_3 +$$
$$(a_{11}c_3^2 + a_{13}c_1c_3 + a_{33}c_1^2)\ x_3^2=0$$

The condition that this shall be a perfect square is (assuming c_1 not 0) $c_1a_{23} + a_{13}c_2 + c_3a_{12}=0$ which is seen to be the condition that the line $c_1x_1 + c_2x_2 + c_3x_3=0$ pass through the point (a_{23},a_{13},a_{12}). If $c_1=0$ either c_2 or c_3 is not zero and x_2 or x_3 can be eliminated.

If an *outside point* of a conic be defined as any point of intersection of tangents to the conic, it follows that in PG($2,2^n$) a conic has but one outside point. Hence the point of concurrence of tangents to a conic will be referred to as the outside point of the conic.

Corollary 1. Every line through the outside point of a conic is a tangent to the conic.

Corollary 2. Through any point in PG ($2,2^n$) other than the outside point of a given conic passes one and but one tangent to the conic.

Corollary 3. In PG($2,2^n$) the condition for degeneracy of a conic is that the coordinates of its outside point satisfy its equation.

For the general conic $F(x_1,x_2,x_3)=0$ the condition is

$$a_{11}a_{23}^2+a_{12}a_{23}a_{13}+a_{33}a_{12}^2=0$$

Corollary 4. In PG($2,2^n$) six and but six points can be chosen such that no three of the set are collinear.

For, any five points no three of which are collinear determine a conic which together with its outside point constitutes a set of six points no three of which are collinear. The existence of any other point not collinear with any two of these contradicts Corollary 2 above.

Corollary 5. In PG($2,2^2$) the diagonal line of the complete quadrangle of any four points of a conic is the tangent of the fifth point.

For, in PG($2,2^2$) there are five points to the line and hence the diagonal line contains two points other than the diagonal points. These two points together with the four points determining the quadrangle form a set of six points no three of which are collinear. It follows, then, from Corollary 4 that the two points on the diagonal line are the fifth point and the outside point respectively of any conic passing through the points of the quadrangle.

§ 3. *Types of Collineations in PG($2,2^n$).*

Transformations on the line. To determine the one-dimensional transformations in PG($2,2^n$) it is sufficient to consider the transformations of points on the line $x_3=0$ and any point on it can be represented by two coordinates. Hence the general projective transformation of points on a line in PG($2,2^n$) may be written

$$T\colon\ \rho x_i'=a_{i1}x_1+a_{i2}x_2, \qquad (i=1,2),$$

where the determinant $\Delta=|a_{ij}|$ of the transformation is not zero and the domain for variables and coefficients is the GF(2^n). In the usual manner we put $x_i'=x_i$, (i=1,2) in order to determine the fixed elements and consider the characteristic equation

$$\rho^2+(a_{11}+a_{22})\rho+\Delta=o \qquad (1)$$

which expresses the condition for consistency.

According as (1) has one, two, or no roots in the GF(2^n) T has one, two, or no fixed points in PG(2,2^n) and is designated correspondingly as *parabolic, hyperbolic,* or *elliptic*. In PG(2,2^n) there are 2^n (2^n–1) equations of the form (1) since Δ is not zero and there are 2^n–1 marks other than zero in the GF(2^n). One half of these equations have both roots in the GF(2^n) and the other half have no roots in the GF(2^n). Of those having roots in the GF(2^n), 2^n–1 have coincident and (2^n–1)(2^{n-1}–1) have distinct roots. The necessary and sufficient condition that (1) shall have coincident roots is that $a_{11}\equiv a_{22}$(mod.2). When $a_{11}=a_{22}$ is substituted in T it is found that T^2 becomes the identical collineation and hence every parabolic transformation in PG(2,2^n) is of period two. An hyperbolic transformation permutes 2^n–1 points and hence its period is 2^n–1 or some factor of 2^n–1. Similarly the period of an elliptic transformation must be 2^n+1 or some factor of 2^n+1.

Suppose (1) to be irreducible in the GF(2^n). From the theory of Galois fields we know that its roots then are marks ρ_1, $\rho_1^{2^n}$ of the GF(2^{2n}) conjugate with respect to the GF(2^n). Substituting these values in T we find the invariant points of T to be (a_{12},$a_{11}+\rho_1$) and (a_{12},$a_{11}+\rho_1^{2^n}$). There are, therefore, on the line 2^n(2^n– 1) points of PG(2,2^{2n}) (referred to as "imaginary" points) arranged in 2^{n-1} (2^n–1) pairs of conjugate points which figure as the double points of the elliptic transformations. Hence in PG(2,2^2) there are six such pairs on each line.

As in ordinary projective geometry it can be proved that any three points on the line in PG(2,2^n) can be transformed into any other three points of the line and that if three points of the line are fixed all points of the line are fixed. From this it follows that the number of parabolic transformations is 2^{2n}– 1, of hyperbolic transformations is 2^{n-1}(2^n–2)(2^n+1) and of elliptic transformations is 2^{2n}(2^n–1)/2. The total number of collineations on the line is the total number of distinct transformations T having coefficients in the GF(2^n) and determinant not zero. This is determined to be 2^n(2^{2n}–1).

In PG(2,2^2) according to the above there ore 60 transformations on the line and of these 15 are parabolic, 20 hyperbolic and 24 elliptic. Since the total group is of order 60 and can be exhibited as permutations of the five points of the line, it must be the alternating group on five symbols and hence its subgroups are well known. They will, however, be enumerated later in determining the groups which leave a line invariant.

Transformations in the plane. The general linear homogenous transformation in PG(2,2^n) may be written

$$T_1:\rho x_i'=a_{i1}x_1+a_{i2}x_2+a_{i3}x_3, \qquad i=1,2,3,$$

where the determinant $\Delta=|a_{ij}|$ of the transformation is not zero and the domain for the coefficients is the GF(2^n). To determine the invariant points we set $x'_i=x_i$, (i=1,2,3) and obtain the characteristic cubic

$$\rho^3 + (a_{11}+a_{22}+a_{33})\ \rho^2 + (A_{11}+A_{22}+A_{33})\ \rho+\Delta=o \qquad (2)$$

where A_{ij} is the co-factor of a_{ij} in $\Delta=|a_{ij}|$. Since there are 2^n marks in the GF(2^n) there are $2^{2n}(2^n-1)$ equations of the form (2) belonging to the GF(2^n). Of these 2^n-1 have all three roots coincident, $(2^n-1)\ (2^n-2)$ two roots coincident and the other distinct, $(2^n-1)(2^n-2(2^n-3)/3!$ all three roots distinct, and $2^n(2^n-1)^2/2$ one root in the GF(2^n) and two roots in the GF(2^{2n}) but conjugate with respect to the GF(2^n). These last are made up of the products of the 2^n-1 linear factors in GF(2^n) with the $2^n(2^{n-1})/2$ irreducible quadratics which appeared as characterictic equations of transformations on the line. There are then $2^n(2^n-1)(2^{n+1}-1)/3$ cubics (2) having roots in GF(2^n) or GF(2^{2n}). The remaining $2^n(2^{2n}-1)/3$ are irreducible in the GF(2^n) but from the theory of Galois fields* it follows that their roots are marks of the GF(2^{3n}) conjugate with respect to the GF(2^n). If, therefore, λ be a root of an irreducible cubic belonging to the GF(2^n) its other roots are λ^{2^n} and $\lambda^{2^{2n}}$, where λ is a mark of the GF(2^{3n}). If we put $x_i'=x_i$, (i=1,2,3) in T_1 and substitute λ for ρ we obtain

$$x_1:x_2:x_3\equiv(A_{11}+a_{22}\lambda+a_{33}\lambda+\lambda^2):(A_{12}+a_{21}\lambda):(A_{13}+a_{31}\lambda)$$

as the corresponding invariant point. The other invariant points are then necessarily the points obtained by substituting for λ in this expression λ^{2^n} and $\lambda^{2^{2n}}$ respectively. Hence every transformation T_1 whose characteristic equation (2) is irreducible in the GF (2^n) leaves invariant a triangle in PG($2,2^{3n}$) which will be designated as an *imaginary triangle* to indicate that it is not in PG($2,2^n$).

Corresponding to the three cases in which the characteristic equation (2) has three distinct roots there are then three types of transformations having for invariant figure a triangle. These will be designated as type I_0, type I_1, type I_3 according as the invariant triangle has none, one, or three of its vertices in PG($2,2^n$).

If the equation (2) has two roots coincident the corresponding collineation is designated type II and leaves invariant two points and by duality two lines. Its invariant figure has two points on one line and two lines on one point. If the three roots of (2) coincide the corresponding transformation leaves invariant a lineal element and is called type III. Two special cases of these will be classified as separate types because of their importance. A transformation other than the identity which leaves invariant all points of a line l called the *axis* and all lines through a point P called the center, is called a *homology* or type IV. Such transformations appear among the powers of those of types I_1 and II. A transformation, other than identity, which leaves invariant all points of a line l and all lines through a point P on l is called an elation or type V. The point P and the line

* Cf. DICKSON, l. c., p 21 and p. 53.

l are called the *center* and *axis*, respectively, of the elation. Such transformations appear among the powers of those of types II and III.

The invariant figures of the different types are shown in Fig. 2. Fixed lines which are imaginary are dotted and fixed points which are imaginary are left open.

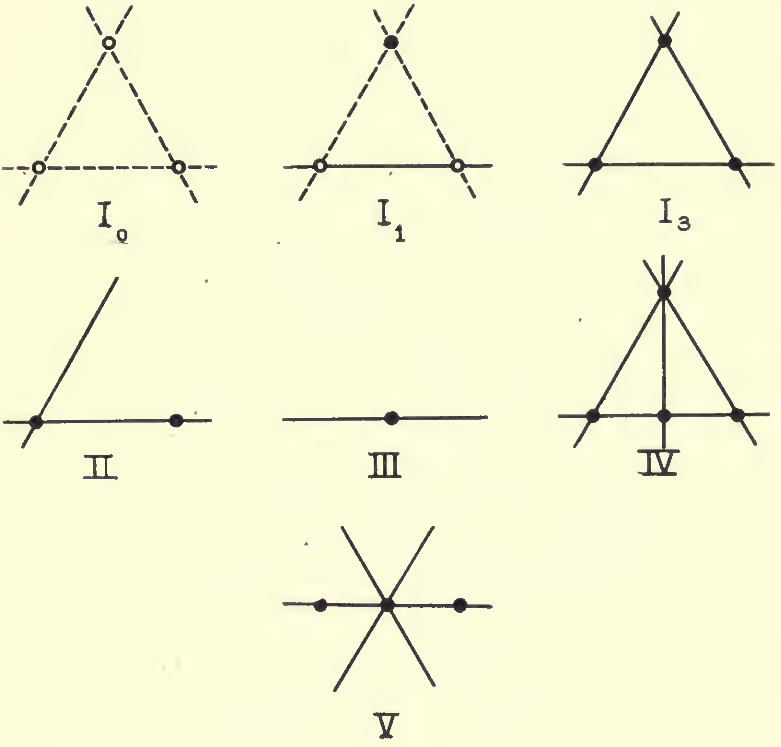

FIG. 2.

The following formulae* for the number of transformations of each type in PG($2,2^n$) are readily obtained:

---

* Cf. Dickson, l. c., pp. 237-239.

$N_{I_0}=2^{4n}(2^{2n}-1)^2/3$
$N_{I_1}=2^{4n}(2^{3n}-1)(2^n-1)/2$
$N_{I_3}=2^{3n}(2^{2n}+2^n+1)(2^n+1)(2^n-2)(2^n-3)/6$
$N_{II}=2^{2n}(2^{3n}-1)(2^n+1)(2^n-2)$
$N_{III}=2^n(2^{3n}-1)(2^{2n}-1)$
$N_{IV}=2^{2n}(2^{2n}+2^n+1)(2^n-2)$
$N_V=(2^{3n}-1)(2^n+1)$
Identity$=1$

Total$=2^{3n}(2^{3n}-1)(2^{2n}-1)$

According to these formulae the order of the total group of $PG(2,2^2)$ is 60480 distributed as follows:

$N_{I_0}=19200$
$N_{I_1}=24192$
$N_{I_3}=2240$
$N_{II}=10080$
$N_{III}=3780$
$N_{IV}=672$
$N_V=315$
Identity$=1$

Total$=60480$

The group of all projective transformations in $PG(2,2^2)$ will be designated as G_{60480}.

§4. *Cyclic Groups in* PG$(2,2^2)$.

We wish to determine in detail the path-curves and periodicity of each of the types in $PG(2,2^2)$. In so doing we shall at the same time determine all of the cyclic subgroups.

Type I_0 Consider the collineation

$$T_0:\quad \begin{aligned} \rho x_1' &= ix_1+ix_3 \\ \rho x_2' &= x_1+x_2+x_3 \\ \rho x_3' &= i^2x_1+i^2x_2+x_3 \end{aligned}$$

The determinant of T_0 is $\Delta=i^2$. Its characteristic equation $\rho^3+i\rho^2+i^2\rho+i^2=0$ is irreducible in the $GF(2^2)$ and has for roots v^{47},v^{59},v^{62} where v is a primitive root of the $GF(2^6)$* and hence $v^{21}=i$, $v^{63}=1$. The invariant points of T_0 are, therefore, $A_0\equiv(1,v^9,v^7)$, $B_0\equiv(1,v^{36},v^{28})$ and $C_0\equiv(1,v^{18},v^{49})$. T_0 is of period 21 and permutes the points of $PG(2,2^2)$ in the order of their numbering in the table of alignment. There is, accordingly, some power of T_0 which will transform any given point of $PG(2,2^2)$ into any other given point of $PG(2,2^2)$. To show that every collineation of type I_0 in $PG(2,2^2)$ is conjugate to some

* For Galois field tables, see an article by W. H. BUSSEY, in the *Bull. Amer. Math. Soc.*, Vol. XI, p. 27.

power of T_0 it is only necessary to show that any triangle $A\equiv(\lambda_1,\lambda_2,\lambda_3)$, $B\equiv(\lambda_1^4,\lambda_2^4,\lambda_3^4)$, $C\equiv(\lambda_1^{16},\lambda_2^{16},\lambda_3^{16})$ where $\lambda_1,\lambda_2,\lambda_3$ are any three marks in the GF(2^6) linearly independent with respect to the GF(2^2), can be transformed into A_0,B_0,C_0, respectively, by a transformation in PG($2,2^n$). The condition that $\lambda_1,\lambda_2,\lambda_3$ be linearly independent with respect to the GF(2^2) is necessary because it will be observed that if the coordinates of the point A satisfied the equation $a_1x_1+a_2x_2+a_3x_3=o$ those of B and C would also satisfy the equation

$$(a_1x_1+a_2x_2+a_3x_3)^{2^2}\equiv(a_1x_1+a_2x_2+a_3x_3)^{2^4}\equiv(a_1x_1+a_2x_2+a_3x_3)\equiv o,$$

and any transformation leaving A, B and C invariant would leave invariant a line of PG($2,2^2$) and therefore not be of type I_0.

The conditions that $T=\rho x_i'=\sum_{j=1}^{j=3}a_{ij}x_j$, $(i=1,2,3)$, $|a_{ij}|$ not o, where every a_{ij} is in the GF(2^2) shall transform A into A_0 are

$$\left.\begin{array}{l}a_{31}\lambda_1+a_{32}\lambda_2+a_{33}\lambda_3=v^7\,(a_{11}\lambda_1+a_{12}\lambda_2+a_{13}\lambda_3)\\ a_{31}\lambda_1+a_{32}\lambda_2+a_{33}\lambda_3=v^{61}\,(a_{21}\lambda_1+a_{22}\lambda_2+a_{23}\lambda_3)\end{array}\right\}\ldots\ldots\ldots\ldots\ldots(4)$$

which raised to the 2^2 and 2^4 powers, are seen to be the conditions that T transform B to B_0 and C and C_0. In (4) we may assign a_{31}, a_{32} and a_{33} arbitrarily in the GF(2^2) provided not all are taken as zero. We then have $a_{31}\lambda_1+a_{32}\lambda_2+a_{33}\lambda_3=v^k$ some mark other than zero of the GF(2^6). Since $\lambda_1,\lambda_2,\lambda_3$ are three marks of the GF(2^6) linearly independent with respect to the GF(2^2) it is possible to choose a_{11},a_{12} and a_{13} within the GF(2^2) such that $a_{11}\lambda_1+a_{12}\lambda_2+a_{13}\lambda_3$ is any mark of the GF(2^6.)* Accordingly we take a_{11},a_{12} and a_{13} such that $a_{11}\lambda_1+a_{12}\lambda_2+a_{13}\lambda_3=v^{k-7}$ and similarly a_{21}, a_{22} and a_{23} such that $a_{11}\lambda_1+a_{22}\lambda_2+a_{33}\lambda_3=v^{k-61}$. The desired transformation T is thereby determined within the GF(2^2). Moreover the determinant of T is not zero since

$$\begin{array}{l}a_{11}\lambda_1+a_{12}\lambda_2+a_{13}\lambda_3=v^{k-7}\\ a_{21}\lambda_1+a_{22}\lambda_2+a_{23}\lambda_3=v^{k-61}.\\ a_{31}\lambda_1+a_{32}\lambda_2+a_{33}\lambda_3=v^{k}\end{array}$$

form a set of simultaneous non-homogeneous equations in λ_1,λ_2 and λ_3.

The 21 powers of T_0 form a cyclic subgroup of G_{60480} and since the triangle ABC can be chosen in $2^6(2^4-1)(2^2-1)/3$ different ways there are 960 such conjugate cyclic subgroups in G_{60480}.

Since the determinant of T_0 is i^2 the determinant of T_0^2 is $i^4=i$ and the determinant of T_0^3 is 1. The powers of T_0, then, which are also powers of T_0^3 and no others are of determinant unity. The group G_{21}(cyc. I_0) consisting of the 21 powers of T_0 contains accordingly a self-conjugate subgroup of order 7 consisting of the 7 powers of T_0^3. G_{60480} must contain 960 such cyclic subgroups.

Again, T_0^7 is of period 3. Hence, G_{21} (cyc. I_0) contains a cyclic subgroup of order 3 which must also be self-conjugate since no others powers of T_0 than powers of T_0^7 are of period 3. G_{60480} must contain 960 such conjugate subgroups.

* Cf. DICKSON, l. c., p. 49

T_0^7 must permute all points of $PG(2,2^2)$ in triangles since if it permuted any three collinear points among themselves it would leave invariant the line joining them. It will be seen later (in discussing the simple group G_{168}) that a transformation of type I_0 of period 7 permutes among themselves seven points so related that for every four of the points which are no three collinear the other three are the diagonal points of their complete quadrangle.

Type I_1. The collineation

$$T_1: \begin{array}{l} \rho x_1' = x_1 \\ \rho x_2' = x_3 \\ \rho x_3' = x_2 + x_3 \end{array}$$

has the characteristic equation $(\rho+1)(\rho^2+\rho+1)=0$ whose roots are 1, u and u^4 where u is a primitive root in the $GF(2^4)$ and hence $u^5 \equiv i$ and $u^{15} \equiv 1$. The invariant points of T_1 are $A_1 \equiv (1,0,0)$, $B_1 \equiv (0,1,u)$, $C_1 \equiv (0,1,u^4)$, and T_1 is therefore of type I_1 with A_1 for center (or invariant real point) and $x_1=0$ for axis ,or invariant real line). T_1 is of period 15 and T_1^5 and T_1^{10} are homologies. Accordingly, the group G_{15} (cyc. I_1) consisting of the 15 powers of T_1 contains a self-conjugate cyclic subgroup of order 3 containing two homologies and the identity. Since the determinant of T_1 is i, $T_1^3, T_1^6, T_1^9, T_1^{12}$, T_1^{15} and no other powers have determinant unity and are of period 5 with the exception of $T_1^{15} \equiv I$. G_{15}(cyc. I_1) therefore contains a self-conjugate cyclic subgroup of order 5 consisting of these transformations.

T_1^3, which is of period 5, permutes the lines through A_1 in cyclic order and hence a point P_1 not on the axis has 4 other conjugates P_2, P_3, P_4, P_5 such that no two of the points P_1, P_2, P_3, P_4, P_5 are collinear with A_1. Moreover, no three of the points P_1, P_2, P_3, P_4, P_5, can be collinear, for if they were the line containing them would be invariant under T_1^3. Hence, P_1, P_2, P_3, P_4, P_5 form a point conic having A_1 for outside point. Evidently T_1^3 leaves invariant three such point conics having A_1 for outside point and by duality three line conics having $x_1=0$ for outside line.

Every collineation T_1' of type I_1 in $PG(2,2^2)$ is conjugate to T_1 or some one of its powers since there is in $PG(2,2^2)$ a transformation S transforming any point P′ and line l' into $(1,0,0)$ and $x_1=0$ respectively and a transformation S_1, leaving the point $(1,0,0)$ fixed and changing any pair of conjugate imaginary points on $x_1=0$ into the pair $(0,1,u^4)$. The collineation $(SS_1)T_1'(SS_1)^{-1}$ must then be some power of T_1.

In discussing the one-dimensional transformations it was shown that in $PG(2,2^2)$ there are six pairs of conjugate imaginary points on each line. Hence there are in $PG(2,2^2)$ $21 \cdot 16 \cdot 6 = 2016$ conjugate groups G_{15}(cyc.I_1) each containing a cyclic self-conjugate subgroup of order 5 consisting of the transformations of period 5, and a cyclic self-conjugate subgroup of order 3 consisting of the homologies. There are 2016 of the cyclic subgroups of order 5 but only $21 \cdot 16 = 336$ of the subgroups of order 3 since the same subgroup of order 3 appears with every G_{15}(cyc.I_1) which leaves invariant a given center and axis.

Type I_3. If T_3 be a transformation of type I_3 it leaves invariant a real triangle A,B,C. T_3 is fully determined by its invariant triangle and the transformation of a point P into a point P′ provided the points A,B,C,P and P′ are no three collinear. Hence, (Theorem 4, Cor. 4) there are two choices for P′ for a given point P. Accordingly T_3 is of period three and permutes the 9 points not on the sides of its invariant triangle in three triangles. It should be noted that any one of these triangles together with the points A,B,C form a set of six points no three of which are collinear and therefore constitute in six different ways a point conic and its outside point. Also that any one of these triangles and two of the points A,B,C form a point conic left invariant by T_3 and hence that T_3 leaves invariant 9 different non-degenerate point conics. If A,B,C be taken as the triangle of reference the two transformations of type I_3 which leave it invariant are

$$T_3:\ \begin{aligned}\rho x_1'&=x_1\\ \rho x_2'&=i^2x_2\\ \rho x_3'&=ix_3\end{aligned} \quad\text{and}\quad T_3^2:\ \begin{aligned}\rho x_1'&=x_1\\ \rho x_2'&=ix_2\\ \rho x_3'&=i^2x_3\end{aligned}$$

Since any triangle can be transformed into any other triangle by a collineation within the PG($2,2^2$) it follows that every collineation of type I_3 is conjugate to T_3 or T_3^2. Since $21\cdot20\cdot16/3!=1120$ different triangles can be chosen in PG ($2,2^2$) there are 1120 conjugate cyclic groups G_3 (cyc. I_3).

Type II. A collineation T_2 of type II leaves invariant two real points A,B, and a real line l (distinct from AB) through one of the points, say A. Two lines fixed through A make the transformation of lines through A of period three. One line fixed through B makes the transformation of lines through B of period two.

T_2 is therefore of period 6, but only T_2 and $T_2^5=T_2^{-1}$ are of type II. T_2^2 and T_2^4 are homologies with B for center and l for axis and T_2^3 is an elation with A for center and AB for axis.

If we select A as the point $(0,0,1)$ B as the point $(1,0,0)$ and the line l as the line $x_1=0$ we find that any point P not on AB or l can be transformed into any other point P′ not on AB, l, PA or PB Taking P and P′ as $(1,1,1)$ and $(i,1,0)$ T_2 is determined as

$$T_2:\ \begin{aligned}\rho x_1'&=ix_1\\ \rho x_2'&=x_2\\ \rho x_3'&=x_2+x_3\end{aligned}$$

On the line $x_1=0$ T_2 interchanges the points $(0,1,0)$ and $(0,1,1)$. It is easily seen that T_2 or T_2^{-1} is conjugate to any other transformation T_2' of type II in PG($2,2^2$).

Since the invariant figure can be chosen in $21\cdot10\cdot8=1680$ different ways and for a given point P_1 on l there are three choices for P_1' it follows that G_{60480} contains 5040 cyclic groups G_6 (cyc.II) each containing a self-conjugate cyclic subgroup of order two consisting of T_1^3 (an elation) and the identity, and a self-conjugate cyclic subgroup of order three consisting of T_2^2 and T_2^4 (homologies) and the identity. It is to be noted, however, that each subgroup of order two is common to 16 (since there are 4 choices for B on AB and 4 choices for l through A) different groups G_6 (cyc. II) and hence that there are but 315 such subgroups. Also that each subgroup of order three is common to 15 different groups G_6 (cyc. II) (since there are 5 choices for A on l and 3 choices for the pairing of lines through B in each case) and hence there are but 336 different such subgroups of order three.

Type III. A transformation T of type III leaves invariant a line l and a point A on l. Since one line through A is fixed the transformation of lines through A is parabolic. T^2 is therefore an elation and T^4 must be the identical transformation. T is consequently of period 4 and permutes four points, no one on l and no three of which are collinear, in cyclic order. Since the transformation of four points no three of which are collinear into four such points fully determines a projective transformation it follows that a transformation T of type III is fully determined by any four such points which T permutes in cyclic order.

The collineation of type III determined by permuting the four points $(1,0,0)$, $(1,1,0)$, $(1,0,1)$, $(1,1,1)$, no three of which are collinear, in cyclic order as named is

$$T:\quad \begin{aligned} \rho x_1' &= x_1 \\ \rho x_2' &= x_1 + x_2 \\ \rho x_3' &= x_2 + x_3 \end{aligned}$$

T leaves invariant the point $(0,0,1)$ and the line $x_1=0$. It is readily seen that in PG$(2,2^2)$ every collineation of type III is conjugate to either T or T^3.

Four noncollinear points A,B,C,D can be chosen in $21\cdot 20\cdot 16:9/4!=2520$ different ways. Each cyclic order determines a transformation of type III not a power of any determined by any other cyclic order and each transformation of type III permutes in cyclic order the points of four different quadrangles. It follows therefore that there are $2520\cdot 3/4=1890$ cyclic groups G_4 (cyc. III) in G_{60480} each containing a self-conjugate cyclic subgroup of order two. It is to be noted that a subgroup of order two is common to 6 different groups G_4 (cyc. III) and hence that there are but 315 such groups.

Type IV. Homologies. The homologies in PG$(2,2^2)$ have appeared as the 336 cyclic subgroups of the 2016 G_{15} (cyc. I_1) and the 5040 G_6 (cyc. II). It was shown that the 336 cyclic G_3 (cyc. IV) were conjugate under the group G_{60480}. A homology, as has been seen, is of period three and its path-curves are the straight lines through the center. The homology

$$T:\quad \begin{aligned} \rho x_1' &= i x_1 \\ \rho x_2' &= x_2 \\ \rho x_3' &= x_3 \end{aligned}$$

may be taken as a canonical form.

Type V. Elations. The elations have appeared as 315 conjugate cyclic subgroups of order two in both the 5040 groups G_6 (cyc. II) and the 1890 groups G_4 (cyc. III). Each elation is of period 2 and its path-curves are the straight lines through the center. The elation

$$T:\quad \begin{aligned} \rho x_1' &= x_1 \\ \rho x_2' &= x_2 \\ \rho x_3' &= x_1 + x_3 \end{aligned}$$

which has for center the point $(0,0,1)$ and for axis the line $x_1=0$ may be taken as a canonical form.

§5. *The Group of Determinant Unity* —G_{20160}.

THEOREM 5. *In PG $(2,2^2)$ every group G of order N which contains collineations of determinant not unity contains exactly* N/3 *collineations of determinant unity.*

Proof. It is obvious that the collineations of determinant unity in G form a self-conjugate subgroup G_n. Suppose n greater than N/3. If T be any collineation in G but not in G_n the products of T and T^2 by the n collineations in G_n are 2n distinct collineations and G would contain $3n>N$ distinct collineations which is contrary to hypothesis. Suppose n to be less than N/3. G must then contain $m>N/3$ collineations of determinant d where d is either i or i^2. If T be any collineation in G of determinant d^2 the products of the m collineations of determinant d by T are $m>N/3$ distinct collineations of determinant unity in G, contrary to supposition. Since n is neither less nor greater than N/3 it follows that $n=N/3$.

The Group of Determinant Unity. By Theorem 5 the group G_{60480} has a self-conjugate subgroup of determinant unity of order $60480/3=20160$. In §4 it was shown that all collineations of types I_3,III, V and those of type I_0 of period 7 and type I of period 5 were of determinant unity. Hence the Group_{20160} of determinant unity contains the following collineations:

The identical collineation	1
All collineations of type I_3	2240
Those collineations of type I_1 which are of period 5, (1-3 of the total number)	8064
Those collineations of type I_0 which are of period 7, (3-10 of the total number)	5760
All collineations of type III	3780
All collineations of type V	315
	20160

The group will be designated as above by G_{20160}. It has been proved that in any PG(k,p^n) the group of all collineations of determinant unity is the maximal simple subgroup of all collineations in the PG(k,p^n).*

THEOREM 6. *Every collineation in G_{20160} can be obtained as a product of elations.*

Proof. Let T be any collineation of type I_3 determined by the equation

* Cf. VEBLEN and BUSSEY, l. c., p. 253 and DICKSON, l. c., p. 87.

$T(A_1A_2A_3A_4)=A_1A_2A_3A_5$ where no three of the points A_1,A_2,A_3,A_4,A_5 are collinear. Two elations E_1 and E_2 are determined by the following equations:

$$E_1(A_1A_2A_3A_4) = A_1A_3A_2A_4$$
$$E_2(A_2A_3A_4A_5)=A_3A_2A_5A_4$$

such that their product $E_2E_1=T$. That E_1 and E_2 are elations follows from the facts that elations are the only collineations in $PG(2,2^2)$ of period two and that the points A_1,A_2,A_3,A_4,A_5 are no three collinear. Since the five points are no three collinear they form a conic and since E_2 interchanges four of the points by pairs it leaves invariant point by point the diagonal line of the complete quadrangle of the four points. By Corollary 5 of Theorem 4 this line contains the fifth point of the conic. E_2, therefore, leaves A_1 invariant and it is clear that $E_2E_1 = T$. Hence every collineation of type I_3 can be obtained as the product of two elations.

Let T_1 be any collineation of type I_1 of period 5 determined by the equation

$$T_1 \ (A_1A_2A_3A_4) = A_2A_3A_4A_5,$$

where A_1,A_2,A_3,A_4,A_5 are five points no three of which are collinear. Two elations E_1' and E_2' are determined by the equations

$$E_1'(A_1A_2A_4A_5)=A_5A_4A_2A_1,$$
$$E_2'(A_2A_3A_4A_5)=A_5A_4A_3A_2,$$

such that $E_2'E_1'= T_1$. That E_1' is an elation leaving A_3 invariant and that E_2 is an elation leaving A_1 invariant follows by the reasoning given above to show that E_2 was an elation leaving A_1 invariant. Hence every collineation T_1 of type I_1 of period five can be obtained as the product of two elations.

The transformation

$$T_0: \quad \begin{aligned} \rho x_1' &= x_1+x_2 \\ \rho x_2' &= x_1+x_3 \\ \rho x_3' &= x_1+x_2+x_3 \end{aligned}$$

is of type I_0 of period 7. It is found that $T_0= T_1E$ where E is an elation,

$$E: \quad \begin{aligned} \rho x_1' &= x_1 \\ \rho x_2' &= x_2 \\ \rho x_3' &= x_1+x_3 \end{aligned}$$

and T_1 is of type I_1 of period five

$$T_1: \quad \begin{aligned} \rho x_1' &= x_1+x_2 \\ \rho x_2' &= x_3 \\ \rho x_3' &= x_2+x_3 \end{aligned}$$

But since every collineation of type I_1 of period five can be obtained as a product of elations and T_0 or one of its powers is conjugate to every collineation of type I_0 of period seven within the $PG(2,2^2)$ it follows that every collineation of type I_0 of period seven can be obtained as a product of elations.

Let S be any collineation of type III determined by the equation

$$S(A_1A_2A_3A_4)=A_2A_3A_4A_1,$$

where no three of the points A_1,A_2,A_3,A_4 are collinear. Any transformation so determined must be of type III because in $PG(2,2^2)$ transformations of type III and no others are of period four. Then $S=E_2E_1$ where E_1 and E_2 are elations determined by the equations

$$E_1(A_1A_2A_3A_4) = A_1A_4A_3A_2,$$
$$E_2(A_1A_2A_3A_4) = A_2A_1A_4A_3.$$

E_1 and E_2 are again necessarily elations because elations are of period two and the points are no three collinear.

Since every collineation not an elation in G_{20160} must be of type I_3, I_1 (of period 5), I_0 (of period 7), or III the theorem is established.

THEOREM 7. *In PG$(2,2^2)$ if a group G_a of determinant unity be transitive on all points and lines of the plane and contain a single elation it contains all elations.*

Proof. In PG$(2,2^2)$ three and but three elations have the same center and axis since there are but three ways in which the four points other than the center on an invariant line can be paired. The theorem will follow, therefore, if it can be shown that if G_a contain a single elation it must contain elations such that for any given line l and point P on l there are three elations in G_a having P for center and l for axis.

From the transitivity of G_a it follows that G_a must contain transforms of the given elation such that every point in the plane is the center and every line the axis of at least one elation. Also the order of G_a must be a multiple of 21 and therefore G_a must contain a collineation of period 3. Since the only collineations in PG$(2,2^2)$ of determinant unity and period three are of type I_3 it follows that G_a must contain collineations of type I_3 such that every point in the plane is a vertex and every line of the plane is a side of the invariant triangle of at least one collineation of type I_3.

For the given line l, then, there is in G_a an elation E having l for axis and a collineation T of type I_3 having l for an invariant line. Four cases may arise.

(a). P may be the center of F and an invariant point of T.

Since T leaves invariant a point which E transforms they cannot be commutative and hence TET^{-1} and T^2ET^{-2} are the other two elations having P for center and l for axis.

(b). P may be the center of E and not an invariant point of T.

Let A and B be the invariant points on l of T. A must be the center of some elation E_1. If l be not the axis of E_1 we have $E_1TE_1^{-1}=T_1$ a collineation of type I_3 having A and some point B′ different from B on l for invariant points. By transforming T_1 through the power of T which transforms B′ to P the case is reduced to case (a). A similar argument applies to the point B. If neither A nor B be the center of an elation whose axis is not l, by case (a) G_a must contain all elations having A or B for center and l for axis. The three elations E_1, E_2, E_3 having A for center and l for axis form with the identity a group since the product of any two of them is an elation having l for axis and A for center. Similarly the three elations E_1',E_2',E_3 having B for center and l for axis form

a group. The nine products E_iE_j' are all distinct since if $E_iE_j'=E_kE_l'$ it follows that $E_j'E_l'=E_iE_k$ which cannot be true. Moreover, every E_iE_j' is an elation having l for axis since it is of determinant unity, leaves fixed every point of l and can not be the identity. Since the nine elations $E_i E_j'$ are all distinct and have l for axis they include the three elations having P for center and l for axis.
(c). P may not be the center of E and may be an invariant point of T.
P must then be the center of some elation E′ having some other line than l for axis. The transforms of E through E′, T and T^2 give elations such that every point of l other than P is the center of an elation having l for axis. Since the lines through P can be interchanged by pairs in three ways only, the product of some two of these four elations is an elation with P for center and l for axis. This case is thereby reduced to case (a).

(d). P may be neither the center of E nor an invariant point of T. If C, the center of E, be not an invariant point of T by transforming E through T or T^2 (whichever transforms C to P) the case is reduced to case (b). If C, the center of E, be one of the invariants points of T we may transform E through E_1, the elation having P for center and some other line than l for axis, and obtain an elation E_2 having some other point on l for center. If E_2 have for center the other invariant point of T the product EE_2 is an elation E_3 whose center is not one of the invariant points of T. The transforming of E_2 or E_3 through T or T^2 then reduces this case as above to case (b), and completes the proof of the theorem.

Definition of Figure. In PG$(2,2^n)$ a *point figure* is defined as any set of m points where m is any positive integer less than $2^{2n}+2^n+1$. Similarly a *line figure* consists of any m lines. The term *figure* is used to refer to either a point figure or a line figure. A real figure in PG$(2,2^n)$ is a figure all of whose points and lines belong to the PG$(2,2^n)$.

It is obvious that any subgroup of G_{60480} which leaves invariant no real figure is transitive on all points and lines of the plane.

THEOREM 8. *There is no subgroup of G_{20160} which does not leave invariant a real figure on an imaginary triangle.*

Proof. Any such subgroup G_k can contain no elation, for by Theorem 7 if G_k contained a single elation it would contain all elations and hence, by Theorem 6, all collineations in G_{20160}. Also G_k can contain no collineation of type III since the square of a type III is an elation.

Suppose G_k to contain a collineation T_1 of type I_1 and let its center be designated P_1. As was seen in the proof of theorem 7, since G_k is transitive and of determinant unity it contains some collineation T_3 of type I_3 which leaves P_1 invariant. Let l_1 and l_2 be the two lines through P_1 left invariant by T_3. Since T_1 is of period 5 on the lines through P_1 some power of T_1, say T_1^m transforms l_1 to l_2. Let l_3,l_4,l_5 be the lines into which T_1^m transforms l_2,l_3,l_4 respectively. Some power of T_3, say T_3^n, produces among the lines through P_1 the transformation (l_1) (l_2) $(l_3l_4l_5)$. Hence the product $T_1^{2m}T_3^n$ produces among the lines through P_1 the transformation $(l_1l_3)(l_2l_4)(l_5)$

The collineation $T_1^{2m}T_3^n$ leaves invariant the point P_1 and a single line l_5 through P_1. Such a collineation must be of type III or an elation. Hence G_k can contain no collineation of type I_1.

Since the only other collineations of determinant unity are of type I_3 (of period 3) and type I_0 (of period 7) G_k can contain only collineations of these two types. Since $20160=2^6\cdot3^2\cdot5\cdot7$ and the order of G_k must be divisible by 21 the only possible orders for G_k are 21 and 63. But as a consequence of Sylow's Theorem* any group of order 21 or 63 must contain a self-conjugate cyclic subgroup of order 7 since the order of the group can be written in the form $7m(1+7k)$ where 7m is the order of the largest group within which the cyclic subgroup of order 7 is self-conjugate and $1+7k$ is the number of cyclic subgroups of order 7. For order 21 the only possibility is $k=0$ and $m=3$ and for order 63 $k=0$ and $m=9$. In PG(2,2^2) the only possible cyclic group of order 7 is a G_7 which leaves invariant an imaginary triangle F_1. But if the G_7 be self-conjugate within the G_k every collineation in G_k must leave invariant the imaginary triangle F_1. Hence there is no subgroup G_k of G_{20160} which does not leave invariant either a real figure or an imaginary triangle.

THEOREM 9. *There is no subgroup of G_{60480} except G_{20160} which does not leave invariant a real figure or an imaginary triangle.*

Proof. If any subgroup, say G_n, exist it must contain a self-conjugate subgroup H_n of determinant unity which leaves invariant no real figure or imaginary triangle contrary to theorem 8.

§6. *The Group Leaving Invariant an Imaginary Triangle*—G_{63}.

THEOREM 10. *The group of all collineations in* PG(2,2^2) *which leave invariant a given imaginary triangle* F_1 *is of order* 63.

Proof. Let the group be designated G_a. In §4 it was shown that if a collineation leave fixed one vertex of F_1 it leaves fixed every vertex of F_1. Hence every collineation leaving F_1 invariant must either permute the vertices of F_1 in cyclic order or leave each vertex fixed. It was also shown in § 4 that there are exactly 21 collineations—the 21 powers of a type I_0 of period 21—which leave each vortex of an imaginary triangle F_1 fixed. That there can not be more than 21 such collineations follows from the fact that a collineation is fully determined by the leaving fixed of each vertex of an imaginary triangle and the transformation of one real point into another real point.

There can not be more than 21 collineations permuting the vertices $A_1B_1C_1$ of F_1 in a given cyclic order $(A_1B_1C_1)$. For suppose S_1, S_2, S_3,......................S_n to be n such collineations where $n>21$. Then if T be a collineation permuting A_1,B_1,C_1 in the order $(A_1C_1B_1)$ there are within G, $n>21$ collineations TS_1,TS_2, TS_3........... ....,TS_n, distinct from each other and each leaving every vertex A_1,B_1,C_1

* See BURNSIDE, *Theory of Groups*, p. 94.

fixed, contrary to the hypothesis that G_a contains but 21 collineations leaving each vertex of F_1 fixed.

That there exists a group of order 63 leaving F_1 invariant is shown by consideration of the transformations

$$T_0: \begin{array}{l} \rho x_1' = i x_1 + x_3 \\ \rho x_2' = x_1 + i x_2 \\ \rho x_3' = x_2 + i^2 x_3 \end{array} \qquad \text{and } T_3: \begin{array}{l} \rho x_1' = x_1 + x_3 \\ \rho x_2' = x_3 \\ \rho x_3' = x_2 + x_3 \end{array}$$

T_0 is of type I_0 of period 21 and T_3 is of type I_3 of period 3. T_0 leaves fixed each vertex $A_1 \equiv (1, v^{27}, v^{36})$, $B_1 \equiv (1, v^{45}, v^{18})$, $C_1 \equiv (1, v^{54}, v^9)$ [where v is a primitive root of the $GF(2^6)$] of F_1, and T_3 permutes these vertices in the order $(A_1B_1C_1)$. T_0 and T_3, therefore, generate a group of order 63 leaving invariant the imaginary triangle F_1. Since it was shown in §3 that F_1 can be transformed into any other imaginary triangle F_1 by a collineation within the $PG(2,2^2)$, there is a group of order 63 leaving invariant any such triangle.

THEOREM 11. *The only groups in $PG(2,2^2)$ which leave invariant an imaginary triangle* F_1 *are the following:*

A. *Groups leaving each vertex of* F_1 *fixed.*

a. *A cyclic group $G_3(cyc. I_0)$ of collineations of type I_0 of period 3.*

b. *A cyclic group $G_7(cyc. I_0)$ of collineations of type I_0 of period 7.*

c. *A cyclic group $G_{21}(cyc. I_0)$ of collineations of type I_0 of period 21.*

B. *Groups permuting the vertices of F_1.*

a. *A cyclic group $G_3(cyc. I_0)$ of collineations of type I_0 of period 3.*

b. *A cyclic group $G_3(cyc. I_0)$ of collineations of type I_0 of period 3.*

c. *An Abelian group G_9 leaving invariant also a real triangle, and containing besides the identity 6 collineations of type I_0 of period 3 and 2 collineations of type I_3.*

d. *A self-conjugate group G_{21} of determinant unity containing besides the identity 6 collineations of type I_0 of period 7 and 14 collineations of type I_3.*

e. *A group G_{63} of all collineations in $PG(2,2^2)$ which leave F_1 invariant containing besides the identity 6 collineations of type I_0 of period 7, 30 of type I_0 of period 3, 12 of type I_0 of period 21, and, 14 of type I_3.*

Proof. The existence of the group G_{63} of all collineations in $PG(2,2^2)$ leaving F_1 invariant was shown in the proof of the preceding Theorem. The existence of the cyclic subgroups is obvious and the existence of the G_{21} of determinant unity follows from Theorem 5. The group G_9 is a Sylow subgroup and that it is Abelian follows from the fact that its order is the square of a prime.* To establish the Theorem it is only necessary to show further that every subgroup of the G_{63} of all collineations leaving F_1 invariant is one of the kinds enumerated above. The only possible orders for such subgroups are 3, 7, 9, and 21. All subgroups of order 3 or 7 must be among the cyclic groups enumerated above since 3 and 7 are primes. A group of order 9 must be Abelian and by Theorem 5 must contain a $G_3(cyc. I_3)$

* Cf. BURNSIDE, l. c., p. 63.

of determinant unity. But no such G_9 can contain more than one G_3(cyc. I_3); for if T_1 and T_2 be two collineations of type I_3 which do not belong to the same G_3(cyc. I_3) the product of T_1 by the power of T_2 which permutes the vertices of F_1 in inverse order is a collineation of determinant unity leaving each vertex of F_1 fixed and therefore of type I_0 of period 7. Hence every subgroup of G_{63} of order 9 contains a self-conjugate G_3(cyc. I_3) and leaves invariant a real triangle. A subgroup of G_{63} of order 21 must be the direct product of a G_3 and a G_7. Since the G_7 must be a G_7 (cyc. I_0) and the G_3 must be either a G_3(cyc. I_0) or a G_3 (cyc. I_3) every such subgroup must be either a G_{21}(cyc. I_0) or a G_{21} of determinant unity and therefore one of the kinds enumerated in the Theorem.

§ 7. *Invariant Real Figures and Their Groups.*

It has now been shown that every subgroup of the G_{60480} except the self-conjugate G_{20160} of determinant unity leaves invariant a real figure or an imaginary triangle, and every group which leaves invariant an imaginary triangle has been determined. Accordingly we next take up the question of determining what real figures can be the invariant figures of groups in PG($2,2^2$) and what group or groups leave each invariant. In determining these groups it is sufficient to consider point figures; for, since a collineation in the plane is self-dual, corresponding to every group which leaves invariant an n-line figure there is a group of the same order which leaves invariant the dual n-point figure. Abstractly considered the two groups are identical. Furthermore, in PG($2,2^2$) it is sufficient to consider point figures in which the number of points n is less than 11, for if $n \geqq 11$ the point figure consisting of $21-n$ (or some lesser number) can be taken as the invariant figure of the group.

In this section will be determined all groups which leave invariant real point-figures whose points are not all collinear and which leave no point fixed under all transformations of the group. To obtain all such groups it is only necessary to determine for each value of n from $n = 10$ to $n = 3$ all groups which are transitive on all points of the n-point figure; for, if such a group be not transitive on all points of an n-point invariant figure it must appear as a group which is transitive on an m-point figure where $3 \leqq m < n$.

A group which leaves invariant an n-point figure also leaves invariant an associated line figure each line of which contains the same number of points; for, every line containing k of the n points can be transformed by a collineation within the group into some line through each of the other n-k points and hence each of the n points lies on at least one line containing k of the n points. It is, of course, obvious that every transform of a line which contains k of the n points must also contain k of the n points if the n-point figure is invariant under the group. It follows by the same reasoning that through each of the n points there must be the same number of lines. Hence the figure made up of an n-point and its asso ciated m line may be called a *configuration** and represented by the symbol

$$\left|\begin{array}{cc} n & l \\ k & m \end{array}\right|$$

where n is to indicate the total number of points, m the total number of lines, k the number of points on each line and l the number of lines through each point of the configuration. In such a configuration the points and lines are so related that $nl = km$, and $1 + l(k\text{-}1) \not> n$. Also, since in PG$(2,2^2)$ not more than 6 points can be chosen such that no three are collinear (Cor. 4, Theorem 4) if $n > 6$, $k \not< 3$. Since not more than 6 lines can be chosen such that no three are concurrent it follows that if $m > 6$ either $l \not< 3$ or $n \not< 3k$.

By making use of the above relations and the fact that whenever $m < n$ or $21 - m < n$ it follows by duality that the same group must appear as a group leaving invariant a lesser number of points, we find that the possible configurations in PG$(2,2^2)$ reduce to the following:

(a) $\left|\begin{array}{cc} 10 & 3 \\ 3 & 10 \end{array}\right| = F_{10,3}$ (b) $\left|\begin{array}{cc} 9 & 4 \\ 3 & 12 \end{array}\right| = F_{9,3}$ (c) $\left|\begin{array}{cc} 9 & 3 \\ 3 & 9 \end{array}\right| = F_{9,3}'$

(d) $\left|\begin{array}{cc} 8 & 3 \\ 3 & 8 \end{array}\right| = F_{8,3}$ (e) $\left|\begin{array}{cc} 7 & 3 \\ 3 & 7 \end{array}\right| = F_{7,3}$ (f) $\left|\begin{array}{cc} 6 & 5 \\ 2 & 15 \end{array}\right| = F_{6,2}$

(g) $\left|\begin{array}{cc} 6 & 4 \\ 2 & 12 \end{array}\right| = F_{6,2}'$ (h) $\left|\begin{array}{cc} 6 & 3 \\ 2 & 9 \end{array}\right| = F_{6,2}''$ (i) $\left|\begin{array}{cc} 6 & 2 \\ 2 & 6 \end{array}\right| = F_{6,2}'''$

(j) $\left|\begin{array}{cc} 5 & 4 \\ 2 & 10 \end{array}\right| = F_{5,2}$ (k) $\left|\begin{array}{cc} 5 & 2 \\ 2 & 5 \end{array}\right| = F_{5,2}'$ (l) $\left|\begin{array}{cc} 4 & 3 \\ 2 & 6 \end{array}\right| = F_{4,2}$

(m) $\left|\begin{array}{cc} 4 & 2 \\ 2 & 4 \end{array}\right| = F_{4,2}'$ (n) $\left|\begin{array}{cc} 3 & 2 \\ 2 & 3 \end{array}\right| = F_{3,2}$.

It is observed that a group which leaves invariant an $F_{i,j}$ also leaves invariant the figure made up of the remaining points and lines of the plane. This figure will be called the *residual figure* and referred to as $R_{i,j}$.

We will consider these configurations in order.

(a) $\left|\begin{array}{cc} 3 & 10 \\ 10 & 3 \end{array}\right| = F_{10,3}$.

Consider a point P of $F_{10,3}$ and the three lines l_1, l_2, l_3 of

* Cf. VEBLEN and YOUNG, *Projective Geometry*, Vol. I, pp. 38-39.

$F_{10,3}$ which pass through P. On l_1,l_2,l_3 are 7 points of $F_{10,3}$ and hence there are three points P_1,P_2,P_3 of $F_{10,3}$ not on any of the lines l_1,l_2,l_3. This necessitates either that 6 lines l_1,l_2,l_3, PP_1,PP_2,PP_3 pass through P or that two or more of the points P_1,P_2,P_3 are collinear with P. Since neither of these conclusions is allowable under our hypotheses, $F_{10,3}$ is not a possible configuration in PG($2,2^2$).

(b) $\left|\begin{array}{cc} 9 & 4 \\ 3 & 12 \end{array}\right| = F_{9,3}.$

On each line of $F_{9,3}$ must be two points which do not belong to $F_{9,3}$. Let any line of $F_{9,3}$ be chosen as the line $x_1=0$ and the two points on it which do not belong to $F_{9,3}$ as the points **12** $(0,0,1)$* and **17** $(0,1,0)$. Then the points **0** $(0,1,i)$, **10** $(0,1,1)$, and **18** $(0,i,1)$ on $x_1=0$ must be points of $F_{9,3}$. Through **0** passes one and but one line which does not belong to $F_{9,3}$. Let the point of intersection of this line with the similar line through **10** be chosen as the point **4** $(1,0,0)$. Neither of these lines can contain any other points of $F_{9,3}$ than **0** and **10**, respectively, because the other three lines through either point contain six other points of $F_{9,3}$ which added to the three points on $x_1=0$ gives the total nine points of $F_{9,3}$. The point **4** is therefore not a point in $F_{9,3}$. Now no line through **12** which does not pass through **4** can be a line of $F_{9,3}$ since such a line contains at least three points (**12** and its two intersections with the lines from **4** to **0** and **10**, respectively,) not in $F_{9,3}$. A similar argument applies to the point **17**. But every line of $F_{9,3}$ passes through some point of $x_1=0$ and but nine besides $x_1=0$ pass through the points **0**, **10**, **18**. Hence the lines $x_2=0$ and $x_3=0$ are lines of $F_{9,3}$ and the nine points of $F_{9,3}$ lie three by three on the sides of the triangle of reference. On the side $x_2=0$ are the points **6** $(i,0,1)$, **11** $(1,0,1)$, **15** $(1,0,i)$, and on $x_3=0$ are **3** $(1,1,0)$, **7** $(i,1,0)$, **19** $(1,i,0)$. A reference to the table of alignment (p. 3) shows that these nine points are collinear by threes on nine other lines as shown in the accompanying figure (Fig. 3). Since $x_1=0$ was chosen as *any* line

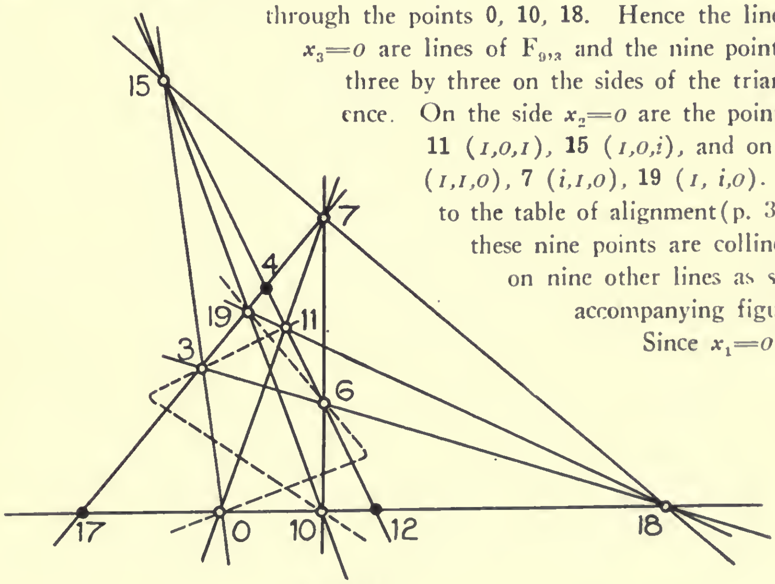

Figure 3

* Numbers printed thus, **12**, **17**, etc., refer to the numbers assigned to points with certain coordinates, as given in the table of alignment on p. 3

in $F_{9,3}$ it follows that through each point not belonging to $F_{9,3}$ pass two lines of $F_{9,3}$ and three lines not belonging to $F_{9,3}$, and each line not belonging to $F_{9,3}$ contains one and but one point of $F_{9,3}$.

Having found that $F_{9,3}$ is a possible configuration in PG$(2,2^2)$ we next proceed to determine what collineations can leave it invariant.

No line of $F_{9,3}$ can be the axis of an elation leaving $F_{9,3}$ invariant, for an elation interchanges all lines not invariant by pairs. Hence not more than one point of $F_{9,3}$ can be invariant under an elation. But since an elation interchanges all points not invariant by pairs at least one of the nine points of $F_{9,3}$ must be invariant under an elation which leaves $F_{9,3}$ invariant. If any point of $F_{9,3}$ is to be the center of such an elation the axis of the elation must be the one line through the point which contains no other point of $F_{9,3}$, that is, the axis must be the line joining the point to the opposite vertex of the triangle of reference. An elation having such a center and axis and interchanging the other two vertices of the triangle of reference must leave $F_{9,3}$ invariant since the nine points of $F_{9,3}$ lie three by three on the sides of the triangle of reference. It is obvious that there exists one and but one such elation* for each point in $F_{9,3}$. Moreover, no point of the residual figure $R_{9,3}$ can be the center of an elation leaving $F_{9,3}$ invariant for through such a point pass two lines of $F_{9,3}$ upon each of which are three points of $F_{9,3}$ which could not be interchanged by pairs. There are, therefore, nine and but nine elations in PG$(2,2^2)$ which leave $F_{9,3}$ invariant.

If T be a transformation of type I_3 which leaves $F_{9,3}$ invariant no point P of $F_{9,3}$ can be a vertex of its invariant triangle; for at least one of the invariant lines through P would have to be a line of $F_{9,3}$ and on that line a point of $F_{9,3}$ would be transformed into a point in $R_{9,3}$. If any point in $R_{9,3}$ can be a vertex of the invariant triangle of T the two lines through it belonging to $F_{9,3}$ must be the two invariant lines through the point, since otherwise at least one of them would be transformed into a line not in $R_{9,3}$. The other two vertices of the triangle must be the two other points on these lines which do not belong to $F_{9,3}$. Since the line joining these two points is a line of $F_{9,3}$ the points of $F_{9,3}$ are three by three on the sides of the triangle and hence T^1 and T^2 leave $F_{9,3}$ invariant. Since but four such invariant triangles can be selected from the twelve points of $R_{9,3}$ there are eight and but eight collineations of type I_5 which leave $F_{9,3}$ invariant.

No transformation of type I_1 of period five or fifteen can leave $F_{9,3}$ invariant on account of its period. A collineation of type I_3 of period three is an homology. If an homology H leave $F_{9,3}$ invariant it can not have a point of $F_{9,3}$ for center since on one of the invariant lines a point of $F_{9,3}$ would be transformed into a point of $R_{9,3}$. If any point P of $R_{9,3}$ can be the center of H, through P pass two and but two lines of $F_{9,3}$ and the axis of H must be the line l joining the two points of $R_{9,3}$ which lie on these two lines. Since l contains the other three points

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* For example, if 0 be chosen as the point, the elation must be $\rho x_1' = x_1, \quad \rho x_2' = i^2 x_3, \quad \rho x_3' = i x_2$

of $F_{9,3}$ a homology having P for center and l for axis must leave $F_{9,3}$ invariant. For each of the twelve points of $R_{9,3}$ there are, then, two and but two homologies leaving $F_{9,3}$ invariant. Accordingly, there are twenty-four homologies leaving $F_{9,3}$ invariant.

If the homology H having P for center and l for axis be multiplied by an elation E leaving $F_{9,3}$ invariant and having some point Q on l for center a collineation T_2 is obtained which transforms $F_{9,3}$ into itself and leaves invariant the points P and Q and the line l. Since T_2 is of determinant not unity and is of period 2 on l and period 3 on the line PQ it must be a collineation of type II. Since there are three and but three choices for the point Q on l there are six and but six collineations of type II having P for center which leave $F_{9,3}$ invariant. But every collineation of type II is of period six and has for its square a homology and for its cube an elation. Hence every collineation of type II which leaves $F_{9,3}$ invariant must be the product of a homology and an elation each leaving $F_{9,3}$ invariant and therefore related as were H and E in obtaining T_2 above. Since the only points which can be the centers of homologies leaving $F_{9,3}$ invariant are the twelve points of $R_{9,3}$ and each homology and its square can be combined with three different elations there are seventy-two and but seventy-two collineations of type II leaving $F_{9,3}$ invariant.

If $F_{9,3}$ can be left invariant by a collineation T_3 of type III, T_3 must have the same center and axis as some elation since T_3^2 is an elation. Taking 0 for center and 0 4 for axis we find that there are six and but six collineations T_3, T_3', T_3'', and their cubes, of type III which leave $F_{9,3}$ invariant and have this center and axis. This corresponds to the fact that there are only three ways in which the four lines of $F_{9,3}$ through 0 can be interchanged by pairs. The transformations T_3, T_3', and T_3'' are

$$T_3:\ \begin{aligned}\rho x_1' &= x_1+i^2x_2+ix_3\\ \rho x_2' &= ix_1+ix_2+ix_3\\ \rho x_3' &= i^2x_1+x_2+ix_3\end{aligned}\qquad T_3':\ \begin{aligned}\rho x_1' &= x_1+ix_2+x_3\\ \rho x_2' &= i^2x_1+ix_2+ix_3\\ \rho x_3' &= x_1+x_2+ix_3\end{aligned}\qquad T_3'':\ \begin{aligned}\rho x_1' &= ix_1+ix_2+x_3\\ \rho x_2' &= ix_1+x_2+ix_3\\ \rho x_3' &= i^2x_1+x_2+x_3\end{aligned}$$

That these collineations leave $F_{9,3}$ invariant is more readily seen when they are written in the form (points of $F_{9,3}$ in italics):

$T_3=(\mathit{0})\ (\mathit{3\ 11\ 15\ 7})\ \ (\mathit{6\ 18\ 19\ 10})\ \ (1\ 16)\ (4\ 14)\ (5\ 12\ 9\ 17)\ (2\ 13\ 8\ 20)$

$T_3'=(\mathit{0})\ (\mathit{3\ 19\ 15\ 6})\ \ (\mathit{7\ 10\ 11\ 18})\ (1\ 14)\ (4\ 16)\ (5\ 13\ 9\ 20)\ (2\ 17\ 8\ 12)$

$T_3''=(\mathit{0})\ (\mathit{3\ 18\ 15\ 10})\ (\mathit{6\ 7\ 19\ 11})\ (1\ 4)\ (14\ 16)\ (5\ 2\ 9\ 8)\ (12\ 20\ 17\ 13)$

Since these collineations are not commutative with the collineations of type I_3 and the elations which change the point 0 into points on the other sides of the triangle of reference it is clear that the total group of collineations leaving $F_{9,3}$ invariant must contain six collineations of type I_3 for each point of $F_{9,3}$ or altogether 54 such collineations. In fact, it is obvious that if any other center and axis than 0 and 0 4 respectively had been selected six and but six collineations of type III leaving $F_{9,3}$ invariant and having that center and axis could have been determined, provided that the center and axis selected were the center and axis of some elation leaving $F_{9,3}$ invariant.

No transformation of type I_0 of period 21 or 7 can leave $F_{9,3}$ invariant on account of its period. If a transformation T_0 be of type I_0 of period three it permutes all points of the plane by triangles. As we have seen, there are four triangles each of which has for vertices points of $R_{9,3}$ and all other points on its sides points belonging to $F_{9,3}$. If T_0 leave $F_{9,3}$ invariant it must transform this set of triangles into themselves, either by permuting the vertices of one of the triangles among themselves or by transforming one triangle wholly into another. Selecting one of these triangles, say **4 12 17**, we determine all the transformations of type I_0 of period three which permute its vertices in the order (4 12 17) and find that there are the six following:

$$\begin{array}{lcccccc} & T_{01} & T_{02} & T_{03} & T_{04} & T_{05} & T_{06} \\ \rho x_1' = & x_2 & x_2 & ix_2 & x_2 & x_2 & i^2x_2 \\ \rho x_2' = & x_3 & ix_3 & x_3 & x_3 & i^2x_3 & x_3 \\ \rho x_3' = & ix_1 & x_1 & x_1 & ix_1 & x_1 & x_1 \end{array}$$

Writing these in the form (points of $F_{9,3}$ in italics)

T_{01}=(4 12 17) (*0 19 11*) (*3 15 10*) (*6 18 7*) (1 2 9) (5 13 8) (14 20 16)
T_{02}=(4 12 17) (*0 7 15*)(*3 11 18*) (*6 10 19*) (1 16 5) (2 14 13) (8 9 20)
T_{03}=(4 12 17)(*0 3 6*) (*7 11 10*) (*15 18 19*) (1 8 14) (2 5 20) (9 13 16)
T_{04}=(4 12 17) (*0 19 15*)(*3 6 10*) (*11 18 7*) (1 14 2) (8 13 9) (5 16 20)
T_{05}=(4 12 17) (*0 3 11*) (*6 18 19*) (*7 15 10*)(1 9 16) (8 20 14) (5 2 13)
T_{06}=(4 12 17) (*0 7 6*) (*3 15 18*)(*11 10 19*)(1 5 8) (2 20 9) (13 14 16)

it is seen that the vertices of no one of the triangles **2 8 16**, **5 9 14**, **1 20 14** are permuted by any of these transformations but that one triangle is transformed wholly into another. Since the squares of these transformations also leave $F_{9,3}$ invariant there are twelve transformations of type I_0 of period three leaving $F_{9,3}$ invariant which permute the vertices of a given one of the four triangles whose vertices are in $R_{9,3}$. There are, therefore, altogether 48 collineations of type I_0 of period three which leave $F_{9,3}$ invariant.

Summarizing, we have in the total group leaving $F_{9,3}$ invariant 9 elations, 8 type I_3's, 54 type III's, 24 homologies, 72 type II's, 48 type I_0's of period 3, and the identity, making a total of 216 collineations. The group is readily identified as the Hessian group G_{216} discovered by C. Jordan in 1878.* Here the 9 points of $F_{9,3}$ represent the 9 points of inflection of each cubic of the pencil

$$\lambda(x_1^3+x_2^3+x_3^3)+\mu x_1x_2x_3=0$$

which is invariant under every collineation of the group. To verify that every point of $F_{9,3}$ lies on every cubic of the pencil it is only necessary to notice that every point of $F_{9,3}$ has one and but one coordinate zero and $i^3=(i^2)^3=1$. Every subgroup of G_{216} leaves $F_{9,3}$ invariant and every group in

* CRELLE, Vol. 84, (1878) pp. 89-215. The group is defined by leaving invariant a pencil of cubics $\lambda F+\mu H=0$ where F is a ternary cubic and H its Hessian covariant.

$PG(2,2^2)$ (except the G_{216} itself) which leaves $F_{9,3}$ invariant must be a subgroup of the $G216$.*

(e) $\begin{vmatrix} 9 & 3 \\ 3 & 9 \end{vmatrix} = F_{9,3}'$. Consider a point P of $F_{9,3}'$ and the three lines l_1, l_2, l_3 of $F_{9,3}'$ which pass through P. On l_1, l_2, l_3 are 7 points of $F_{9,3}'$ and there are, therefore, two points P′ and P″ of $F_{9,3}'$ not on l_1, l_2, or l_3. There are two possibilities only—P′ and P″ are or are not collinear with P. If P′ and P″ are collinear with P the configuration is $F_{9,3}$ except that 3 lines are omitted. If P′ and P″ are not collinear with P there must be two lines through P (namely PP′ and PP″) which contain but two points of $F_{9,3}'$ each. Since P can be transformed into any other point of $F_{9,3}'$ this would necessitate the existence of a configuration of 9 points arranged two points to the line which contradicts the condition that when the number of points exceeds 6 there must be at least 3 to the line. Hence there is but one possible arrangement and that is as the 9 points and 9 of the lines of $F_{9,3}$. Since $F_{9,3}$ includes all lines joining its points it follows that every transformation which permutes the 9 points and 9 of the lines of $F_{9,3}$ also permutes among themselves the other 3 lines of $F_{9,3}$. $F_{9,3}'$ is, therefore, the subgroup G_{54} of order 54 of $F_{9,3}$ which leaves invariant a simple 3-line composed of 3 lines of $F_{9,3}$ so chosen that no two of them meet in a point of $F_{9,3}$.

(d) $\begin{vmatrix} 8 & 3 \\ 3 & 8 \end{vmatrix} = F_{8,3}$. Let any line of $F_{8,3}$ be chosen as the line $x_3=o$. Since but 6 lines of $F_{8,3}$ meet $x_3=o$ in points of $F_{8,3}$ there is one and but one line of $F_{8,3}$ which meets $x_3=o$ in a point of $R_{8,3}$. Let this be chosen as the line $x_2=o$ and let $x_1=o$ be the line joining the two points on $x_3=o$ and $x_2=o$ (other than their point of intersection) which belong to $R_{8,3}$. The points 3, 7, 19 of $x_3=o$ and 6, 11, 15 of $x_2=o$ are then points of $F_{8,3}$. Through each of the points 3, 7, 19 pass 3 lines of $F_{8,3}$ which contain 7 points of $F_{8,3}$. Since the one other point determines but one line with a given point it follows that through each of the points 3, 7, 19 passes one and but one line which contains no other point of $F_{8,3}$. These 3 lines must meet $x_2=o$ in a point of $R_{8,3}$ and hence all pass through the point 12 the intersection of $x_2=o$ and $x_1=o$. Any line through 4 $(1,o,o)$ other than $x_2=o$ and $x_3=o$ intersects these three lines in points of $R_{8,3}$ and hence can contain no points of $F_{8,3}$ except as points of intersection with $x_1=o$. Therefore the other two points of $F_{8,3}$ lie on $x_1=o$. But we know that the 9 points other than vertices on the sides of any triangle in $PG(2,2^2)$ are collinear by threes on 12 lines of which 4 pass through each point. Accordingly, if any one of the points 0, 10, 18 on $x_1=o$ be omitted there remain 8 points collinear by threes on 8 lines. Hence

* The group $G216$ was studied at length by MASCHKE, *Math. Annalen*, Vol. 33 (1890), pp. 324-330, and the geometric properties of the group and its subgroups by NEWSON, *Kansas University Quarterly*, Vol. II, No. 6 (Apr. 1901), pp. 13-22.

$F_{8,3}$ must be $F_{9,3}$ with one point and the 4 lines through that point omitted and its group is the subgroup of $F_{9,3}$ G_{24} of order 24 which leaves invariant a single point.

(e) $\left|\begin{array}{cc} 7 & 3 \\ 3 & 7 \end{array}\right| = F_{7,3}$. Let l be any line of $F_{7,3}$ and P, Q, R the three points on l belonging to $F_{7,3}$. Then there are but four other points A,B,C,D of $F_{7,3}$ not on l and these four points must lie two by two on two lines through each of the points of the complete quadrilateral of the other four points A,B,C,D. Let A,B,C, be taken (See Fig. 1, p. 4) as the vertices $(1,0,0)$,$(0,0,1)$ $(0,1,0)$ respectively of the triangle of reference and D as the point $(1,1,1)$. This determines the coordinates of P,Q,R as $(1,0,1)$,$(0,1,1)$, $(1,1,0)$ respectively. Since these coordinates are in the GF(2) $F_{7,3}$ coincides with PG(2,2) and we can determine the number of collineations of each type by substituting n=1 in the formulae given on p. 10, noting that type I_1 of PG(2,2) is type I_3 of PG$(2,2^2)$. This gives 56 collineations of type I_3 (type I_1 in PG(2,2)), 48 of type I_0 of period 7, 42 of type III, and 21 elations, which, together with the identical transformation make a group G_{168} of order 168. Every collineation in the group is of determinant unity and since it is of degree 7 and order 168 it is recognized as the simple group G_{168} first derived by Klein by the consideration of the transformation of the seventh order of elliptic functions.*

Every group which leaves $F_{7,3}$ invariant must be a subgroup of G_{168} and every subgroup of the G_{168} leaves $F_{7,3}$ invariant†

(f) $\left|\begin{array}{cc} 6 & 5 \\ 2 & 15 \end{array}\right| = F_{6,2}$; (g) $\left|\begin{array}{cc} 6 & 4 \\ 2 & 12 \end{array}\right| = F_{6,2}'$; (h) $\left|\begin{array}{cc} 6 & 3 \\ 2 & 9 \end{array}\right| = F_{6,2}''$;

(i) $\left|\begin{array}{cc} 6 & 2 \\ 2 & 6 \end{array}\right| = F_{6,2}'''$.

Since the six points of $F_{6,2}$ are no three collinear we may choose any three of them for the vertices 4 $(1,0,0)$, 12 $(0,0,1)$, 17 $(0,1,0)$ of the triangle of reference and any point not collinear with any two of these, say $(1,1,i)$ may be taken as the fourth point. Since in PG$(2,2^2)$ the choice of four points no three of which are collinear determines uniquely (Cf. Corollaries 4 and 5 of Theorem 4) the other two points not collinear with any two of them, the fifth and sixth points are necessarily 13 $(i,1,1)$ and 20 $(1,i,1)$. Six points, no three of which are collinear, de-

* *Math. Annalen*, Vol. 14 (1878), p. 438. JORDAN, in determining the finite ternary groups missed both this group and the simple G_{360}. The G_{168} is discussed at some length by BURNSIDE, *Theory of Groups*, pp. 208-209 and 302-305, and in KLEIN-FRICKE'S *Modulfunctionen*.

† For a list of all groups whose degree does not exceed 8, see MILLER, *Amer. Journal of Math.* Vol. 21 (1899), p. 326. The types of substitutions and of subgroups of the G_{168} are given by GORDAN, *Math. Annalen*, Vol. 25, (1885), p. 462.

termine 15 distinct lines. Hence every collineation which leaves invariant the six-point also leaves invariant the associated fifteen-line. From this is follows that the groups leaving $F_{6,2}'$, $F_{6,2}''$, $F_{6,2}'''$, invariant must either be the group leaving $F_{6,2}$ invariant or subgroups of it.

An elation E leaving $F_{6,2}$ invariant can not have more than two points of $F_{6,2}$ on its axis, and since an elation interchanges by pairs all points not on its axis E must interchange by pairs at least four points of $F_{6,2}$. Since any four points no three of which are collinear can be transformed into any four such points by a collineation there exist in PG($2,2^2$) collineations interchanging by pairs any four points of $F_{6,2}$. All such collineations are elations because no other transformations in PG($2,2^2$) are of period two. Moreover, such an elation E leaves invariant each diagonal point of the complete quadrangle of the four points chosen and therefore the other two points on the diagonal line which are not diagonal points. These last two points must be the other two points of $F_{6,2}$ (Cf. proof of Cor. 5, Theorem 4). Since four points no three of which are collinear can be interchanged by pairs in three different ways it follows that there are three elations leaving $F_{6,2}$ invariant for each distinct quadrangle that can be chosen from $F_{6,2}$. There are then $3(6\cdot5\cdot4\cdot3/4!)=45$ elations which leave $F_{6,2}$ invariant.

Since any four points no three of which are collinear can be transformed into any four such points by a collineation there exist in PG($2,2^2$) collineations permuting any four points of $F_{6,2}$ in any given cyclic order. Such a transformation must be of period four and therefore of type III. A collineation T of type III which permutes in cyclic order any four points of $F_{6,2}$ must leave invariant one of the diagonal points of the complete quadrangle of the four points and interchange the other two diagonal points. Since any two points of $F_{6,2}$ lie on the diagonal line of the complete quadrangle of the other four points but are not diagonal points it follows that if T permutes in cyclic order four points of $F_{6,2}$ it interchanges the other two points of $F_{6,2}$. Since any four points, no three of which are collinear, can be permuted in six different cyclic orders it follows that there are $6\ (6\cdot5\cdot4\cdot3/4!)=90$ collineations of type III which leave $F_{6,2}$ invariant.

Since a collineation of type I_1 of period 5 leaves invariant one real point if it leave $F_{6,2}$ invariant its center must be a point of $F_{6,2}$. The other points of $F_{6,2}$ form a conic of which the sixth point (the center) is the outside point. If **4** $(1,0,0)$ be taken as the center and the other five points permuted in the order (**1 13 20 12 17**) the transformation is found to be

$$\mathrm{T}:\quad \begin{aligned} \rho x_1' &= x_1 + i^2 x_2 \\ \rho x_2' &= i^2 x_2 + x_3 \\ \rho x_3' &= x_2 \end{aligned}$$

of type I_1 and period 5. It was shown in § 3 that there are six different pairs of imaginary points on the axis of T determining six transformations of type I_1 with the same center and axis and no one a power of another. These correspond to the six different cyclic orders in which five points of the conic can be permuted and there are, therefore, 6 independent transformations of period 5 (24 altogether) leaving $F_{6,2}$ invariant and having the point 4 for center. Hence there are altogether $6 \cdot 24 = 144$ collineations of type I_1 leaving $F_{6,2}$ invariant.

Since a transformation of type I_3 permutes in cyclic order any set of three points so related to the vertices of its invariant triangle that the six points are no three collinear, any collineation of type I_3 which has three points of $F_{6,2}$ for vertices of its invariant triangle must leave $F_{6,2}$ invariant. Twenty distinct triangles can be chosen from the six points of $F_{6,2}$ and hence 40 collineations of type I_3 leave $F_{6,2}$ invariant in this way. We know, however, that each such collineation permutes the vertices of two triangles not in $F_{6,2}$ either of which may be taken as the invariant triangle of a transformation of type I_3 which permutes the vertices of the two triangles in $F_{6,2}$. For each pair of triangles that can be selected in $F_{6,2}$ there are then four transformations of type I_3 having invariant triangle not in $F_{6,2}$ which leave $F_{6,2}$ invariant. Since ten pairs of triangles can be chosen in $F_{6,2}$ there are 40 collineations of type I_3 which leave $F_{6,2}$ invariant in this way.

It has been shown that the group which leaves $F_{6,2}$ invariant must contain as many as 360 collineations. Since the group can be represented as a substitution group on 6 symbols it must therefore be either the alternating or symmetric group of degree six. But since there is no collineation which holds four of the points of $F_{6,2}$ each fixed and interchanges the other two the group can not be the symmetric group. The group which leaves $F_{6,2}$ invariant is therefore the alternating group on six symbols, shown by WIMAN* to be identical abstractly with the the finite ternary group G_{360} first set up by H. Valentiner.†

The group G_{360} is here characterized not only as a group on six points but since any one of the points can be taken as the outside point of the conic determined by the other five points as a group leaving invariant a system of six conics. It is clear that every subgroup of G_{360} leaves $F_{6,2}$ invariant and every group in $PG(2,2^2)$ which leaves $F_{6,2}$ invariant must be a subgroup of G_{360}.

$$(j)\ \left|\begin{matrix} 5 & 4 \\ 2 & 10 \end{matrix}\right| = F_{5,2}; \qquad (k)\ \left|\begin{matrix} 5 & 2 \\ 2 & 5 \end{matrix}\right| = F_{5,2}'.$$

Since the five points of $F_{5,2}$ are no three collinear they form a conic and the group which leaves $F_{5,2}$ invariant is therefore simply isomorphic with the group of all transformations of points on a line. The group accordingly contains 15 elations, 20 type I_3's and 24 type I_1's of period 5 (Cf. § 3). Since every transformation which leaves a conic invariant must leave its outside point invariant this group is recognized as the subgroup of the G_{360} which leaves a single point fixed. It is here represented both as the group which leaves invariant a conic and the alternating group on five symbols (the five points of $F_{5,2}$). The group which leaves $F_{5,2}'$ invariant must be the subgroup of the G_{60} which leaves the 10 lines of

* *Math. Annalen*, Vol. 47, (1896), p. 531.

† *Kjoeb. Skr.* (5) 5 (1889), p. 64. See *Ency. d. Math., Wiss.*, Vol. I, p. 529. In determining the finite ternary groups, Valentiner, who was apparently unaware of the previous work of Klein and Jordan, missed the G_{360}.

$F_{5,2}$ invariant in two systems of five lines each. It is therefore a group G_{10} of order 10.

(l) $\begin{vmatrix} 4 & 3 \\ 2 & 6 \end{vmatrix} = F_{4,2}$; (m) $\begin{vmatrix} 4 & 2 \\ 2 & 4 \end{vmatrix} = F_{4,2}'$.

The configuration $F_{4,2}$ is a complete quadrangle and since four points can be permuted among themselves in all ways by transformations in the plane the group must be the symmetric group G_{24} of all transformations on the four points. The configuration $F_{4,2}'$ is a simple quadrangle and hence its group is the subgroup G_6 of the G_{24}.

(n) $\begin{vmatrix} 3 & 2 \\ 2 & 3 \end{vmatrix} = F_{3,2}$. The configuration $F_{3,2}$ is the triangle and hence every transformation leaving it invariant must do so in some one of the following three ways: (1) Leave the three vertices each fixed; (2) leave one vertex fixed and interchange the other two; (3) permute the three vertices in cyclic order.

These may be written down at once as follows:

Under (1) there are 2 type I_3's and 6 homologies; under (2) there are 18 type II's and 9 elations; under (3) there are 6 type I_3's and 12 type I_0's of period 3. Including the identity, then, there are 54 collineations in the group.

§ 8. *Subgroups of the Group G_{2880} Which Leaves a Line Invariant.*

All groups which leave invariant a set of collinear points must also leave invariant the line l which contains the points. Since any four lines no three of which are concurrent can be transformed into any four such lines by a projective transformation in PG(2,pⁿ) the order of the group leaving a line fixed is

$$N=(p^{2n}+p^n)(p^{2n})p^{2n}-2p^n+1)=p^{3n}(p^{2n}-1)(p^n-1)$$

For PG(2,2²) this gives N=2880 and accordingly the group will be designated as G_{2880}. Since any line in the plane can be transformed into any other line in the plane by a collineation within the G_{60480} it follows that G_{60480} contains 21 conjugate groups G_{2880}.

Subgroups of G_{2880} which leave a point not on l invariant. In determining the subgroups of G_{2880} we shall first determine all subgroups which leave invariant at least one point not on l and then all subgroups which leave invariant no point not on l. Taking l as the line $x_3=o$ (or the line at infinity) every collineation in the G_{2880} is of the form

$$\mathrm{T}: \begin{matrix} x'=a_1x+a_2y+a \\ y'=b_1x+b_2y+b \end{matrix}$$

where x and y are nonhomogeneous point coordinates. Selecting the point not on l as the origin every collineation in the G_{2880} which leaves it invariant is of the form

$$\mathrm{T}_0: \begin{matrix} x'=a_1x+a_2y \\ y'=b_1x+b_2y \end{matrix}$$

But it has been seen in § 3 that in homogeneous coordinates the group of all transformations of the form T_0 is the group G_{60} of all transformations of points on a line. Since G_{60} is the alternating group on five symbols it contains the following subgroups:*

I. Subgroup leaving all points of the line fixed.
   1 self-conjugate G_1—the identity. $T_1: x'=a_1x, y'=a_1y$.

II. Subgroups leaving invariant two points of the line.
   a. Those leaving each point of the pair fixed.
      10 groups G_3 each conjugate to the G_3 of transformations of the form $T_2: x'=a_1x, y'=b_2y$, a_1, b_2 in the GF(2^2).
   b. Those leaving the pair invariant.
      10 groups G_6 each conjugate to the G_6 of transformations of the form T_0 subject to the restriction that either $a_1=b_2=o$ or $a_2=b_1=o$.
      10 groups G_2 each conjugate to the subgroup of G_6 for which the coefficients are in the GF(2).

III. Subgroups leaving one point of the line invariant.
      15 groups G_2 each conjugate to the G_2 of transformations of the form $T_3: x'=a_1x+a_2y, y'=a_1y$, where a_1, a_2, are in the GF(2^2).
      5 groups G_4 each conjugate to the G_4 of transformations of the form T_3 where a_1, a_2 are in the GF(2^2).
      5 groups G_{12} each conjugate to the G_{12} of all transformations of the form $T_4: x'=a_1x+a_2y, y'=b_2 y$, where $a_1, a_2 b_2$ are in the GF(2^2).

IV. Subgroups leaving invariant a pair of imaginary points on the line.
   a. Those leaving each point of the pair fixed.
      6 groups G_5 each conjugate to the G_5 of all transformations of the form T_0 subject to the condition that

$$a_1^2+ia_2^2+i^2b_1^2+b_2^2+a_1a_2+i^2a_1b_1+i^2a_2b_1+a_2b_2+i^2b_1b_2=o$$

   b. Those leaving the pair invariant.
      6 groups G_{10} each conjugate to the G_{10} of all transformations of the form T_0 subject to the condition that

$$(a_1^2+ia_2^2+i^2b_1^2+b_2^2+a_1a_2+i^2a_1b_1+i^2a_2b_1+a_2b_2+i^2b_1b_2)(a_1+b_1+ia_2)=o$$

If T_0 be taken as a transformation in nonhomogeneous coordinates we have a group G_{60} simply isomorphic with $G_{60}{}^5$ or a group G_{180} triply isomorphic with $G_{60}{}^5$ according as the determinant of the group of transformations of the form T_1 is unity or unrestricted within the GF(2^2). Corresponding to the above groups on the line there are then the following groups in the plane which leave invariant the line l and point (o,o).

I. Subgroups leaving all points of l fixed.
   1. Of determinant unity, 1 self-conjugate G_1.
   2. Of determinant not restricted, 1 self-conjugate G_3.

II. Subgroups leaving invariant a pair of points on l.
   1. Those leaving each point of the pair fixed.
      a. Of determinant unity, 10 conjugate G_3 each leaving a triangle invariant.
      b. Of determinant not restricted, 10 conjugate G_9 each leaving a triangle invariant.

* All groups of degree less than 6 were obtained by Serret.

2. Those leaving the pair of points invariant.
   a. Of determinant unity,
      10 conjugate G_6 each leaving a triangle invariant.
   b. Of determinant unrestricted,
      10 conjugate G_{18} each leaving a triangle invariant.

III.. Subgroups leaving one point of l fixed.

1. Of determinant unity,
   15 conjugate G_2 each leaving invariant a point of lines.
   5 conjugate G_4 each leaving invariant a point of lines.
   5 conjugate G_{12} each leaving invariant the line l, the x-axis and the origin

IV. Subgroups leaving invariant a pair of imaginary points on l.

1. Subgroups leaving each point of the pair fixed.
   6 conjugate G_5 of determinant unity.
   6 conjugate G_{15} of determinant not restricted.
2. Subgroups leaving the pair invariant.
   6 conjugate G_{10} of determinant unity.
   6 conjugate G_{30} of determinant not restricted.

Subgroups of G_{2880} which leave no point not on l invariant. In the discussion which follows the term *translation* will be used to indicate an elation having the line l for axis and the term elation will be used only for an elation whose axis is not l. In determining the subgroups of G_{2880} which leave invariant no point not on l we shall first determine all such subgroups containing no translations and then all such subgroups containing translations.

Let G_n be a subgroup of G_{2880} which leaves no point not on l invariant and contains no translation. If G_n contain an homology H having l for axis and A for center, G_n must contain at least one transform H' of H having some other point than A for center. One of the products $H'H$ or $H'H^2$ is of determinant unity and leaves all points on l fixed. It is therefore a translation. Hence G_n can contain no collineation other than the identity leaving all points of l fixed and, consequently, every such group must be simply isomorphic with the G_{60}.

We have seen (ante p. 30) that there is a group G_{60} leaving invariant a point conic and its outside point. By duality there is a G_{60} leaving invariant a line conic and its outside line. Since the line l is the outside line of 48 different line conics there are 48 such groups G_{60} which leave l invariant.

Every transformation of the form

$$E:\quad \begin{aligned} x' &= x + a \\ y' &= y + b, \end{aligned}$$

is a translation and the group of all transformations of the form E where a and b are marks of the GF(2^2) is a G_{16} leaving every point on l fixed. Unless $a=b=o$ E is of period 2 and hence G_{16} contains 15 cyclic subgroups of order two. If $a=o$ and b be allowed to take on all values in the GF(2^2) or if $b=o$ and a be in the GF(2^2) a group of order 4 is obtained, consisting of all translations leaving fixed all lines through a given point P on l. Such a group will be designated as a G_4(P). If a be allowed to take the value o and but one other value and b be restricted in the same way a group of order 4 is obtained containing besides the identity 3 translations no two of which have the same center. Since such a group leaves invariant a complete quadrangle of which the centers of the three elations are the diagonal points, it will be designated as a G_4(Q). If a be in the

GF(2^2) and b in the GF(2) a group G_8 is obtained leaving invariant a point P on l and interchanging the four lines other than l through P by pairs in a given manner.

The G_{16} is an Abelian (or commutative) group since if E_i and E_j be any two elations in G_{16} $E_jE_i=E_iE_j$. Consequently $E_jE_iE_j^{-1}=E_iE_jE_j^{-1}=E_i$, and every subgroup of G_{16} is self-conjugate within the G_{16}. Also every two translations and their product form (with the identity) a group G_4, for if $E_iE_j=E_k$, $E_i=E_kE_j=E_jE_k$ and $E_j=E_kE_i=E_iE_k$. Hence G_{16} contains $15\cdot 14/6=35$ subgroups G_4.*

We shall next determine the subgroups of G_{2880} which are such that every collineation in the group either leaves invariant a point not on l or is the product of such a collineation and a translation in the group.

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* It may be of interest to note that the G_{16} can be represented as a three-space PG(3,2) by letting the G_2, G_4, G_8, correspond to the points, lines and planes, respectively, of the three-space.

The three-space S_3 *has*	*The Group* G_{16} *has*
15 points, 35 lines, 15 planes, arranged	15 subgroups G_2, 35G_4, 15 G_8, arranged
3 points on each line,	3 G_2 in each G_4,
7 points on each plane,	7 G_2 in each G_8,
7 lines through each point,	7 G_4 containing each G_2,
7 lines in each plane,	7 G_4 contained in each G_8,
3 planes through each line,	3 G_8 containing each G_4,
7 planes through each point.	7 G_8 containing each G_2.

If the three-space S_3 be represented by the notation for a configuration (Cf. Moore, *American Journal of Mathematics*, Vol. 18, pp. 264-303; Veblen and Young, *Projective Geometry*. Vol. I, p 38), the same table exhibits the structure of the group G_{16}.

$G_{16} \diagdown S_3$	G_2 — S_0	G_4 — S_1	G_8 — S_2
G_2 — S_0	15	7	7
G_4 — S_1	3	35	3
G_8 — S_2	7	7	15

In the table, S_0 is a point, S_1 a line, S_2 a plane, and the interpretation is obvious from the parallelism given above.

The translations in any subgroup of G_{2880} form a self-conjugate subgroup G_k. If any group G_n has a system of transitivity S which is also a system of transitivity of its self-conjugate subgroup G_k of translations then every collineation in G_n is either a collineation leaving a point O of S invariant or the product of such a collineation and a translation; for, let O be taken as the origin and let T be any collineation in G_n. If T displaces O it changes O to some point A in S. But since S is a system of transitivity for G_k there is in G_n a translation T_1 changing A to O. Hence $T_1T=T_2$ a collineation in G_n leaving O invariant. From $T_1T=T_2$ we have $T=T_1T_2$. Every such group G_n can be obtained then by extending the groups leaving a point fixed by means of translations. In determining the groups below, S will be used to indicate the system of transitivity common to the group obtained and the extending group of translations. E_1, E_2 and E will be used to indicate the forms of translations as follows:

$$E_1= \begin{matrix} x'=x+a \\ y'=y \end{matrix} \qquad E_2= \begin{matrix} x'=x \\ y'=y+b \end{matrix} \qquad E= \begin{matrix} x'=x+a \\ y'=y+b \end{matrix}$$

The product of $T_0= \begin{matrix} x'=a_1x+a_2y \\ y'=b_1x+b_2y \end{matrix}$ and E is of the form

$$T_0E= \begin{matrix} x'=a_1x+a_2y+a_1a+a_2b \\ y'=b_1x+b_2y+b_1a+b_2b \end{matrix}$$

It should, therefore, be observed that in extending a group of collineations of the form T_0 by E_1, E_2 or E the range of values which can be assumed by the additive constants in the product is dependent upon the coefficients of T_0 as well as those of E_1, E_2 or E. In the work below S will be used to indicate a system of transitivity of the extended group G_n which is also a system of transitivity of the extending group G_k of translations. The groups obtained by such extensions are as follows:

I. *Subgroups leaving each point of l fixed.*
   1. Extensions of G_1 by E_1,E_2,E give groups of translations only.
   2. Extensions of the G_3 of the form T_1: $x'=a_1x, y'=a_1y$, a_1 in GF(2²).
      a. By E_1 gives a G_{12} leaving invariant the x-axis. The points on the x-axis form the system S.
      b. By E_2 gives a similar G_{12} leaving invariant the y-axis.
      c. By E gives a G_{48} leaving invariant no point not on l and no line but l. The system S includes all points not on l.

II. *Subgroups leaving a pair of points on l invariant.*
   1. Subgroups leaving each point of the pair fixed. (10 of each).
      A. Those of determinant unity. Extensions of G_3 of the form T_2: $x'=a_1x$, $y'=b_2y$, a_1 and b_2 in the GF(2²).
      a. By E_1 gives a G_{12} leaving the x-axis invariant. The system S includes all points on the x-axis..
      b. By E_2 gives a similar G_{12} leaving the y-axis invariant.
      c. By E gives a G_{48} leaving invariant no point not on l and no line but l. The system S includes all points not on l.

B. Those of determinant not restricted. Extensions of the G_9 of the form T_2. (10 of each).

a. By E_1 gives a G_{36} leaving the x-axis invariant.
The system S includes all points on the x-axis.

b. By E_2 gives a similar G_{36} leaving the y-axis invariant.

c. By E gives a G_{144} leaving invariant no point on l and no line but l.
The system S includes all points not on l.

2. Subgroups leaving the two points invariant as a pair.

A. Those of determinant unity. (10 of each).
Extensions of the G_2 of T_0 with the form $a_1=b_2=o$ or $a_2=b_1=o$, a_1,a_2,b_1,b_2 being in the GF(2).

a. By E_1 extends by E also giving
For a in GF(2) a G_8 leaving invariant a PG(2,2).
The system S includes four points not on l no three of which are collinear.
For a in the GF(2^2) a G_{32} leaving invariant no point not on l and no line but l. The system S includes all points not on l.

b. By E_2 gives same as by E_1 or E.

Extension of the G_6 of the form T_0 with $a_1=b_2=o$, *or* $a_2=b_1=o$ and a_1,b_1,a_2,b_2 in the GF(2^2).
By E_1 or E_2 extends by E giving a G_{96} leaving invariant no point not on l and no line but l. The system S includes all points not on l.

B. Those of determinant not restricted. (10 of each)
Extension of the G_{48} by E_1 or E_2 extends by E with a and b unrestricted giving a G_{288} leaving invariant no point not on l and no line but l.
The system S includes all points not on l.

III. *Subgroups leaving one point on l fixed.*

1. Those of determinant unity.

A. Extensions of G_2 of form T_4: $x'=a_1x+a_2y$, $y'=a_1y$, a_1,a_2 in GF(2).

a. By E_1 with a in the GF(2) gives a G_4 leaving invariant the x-axis. (15 such groups). The system S is the points on the x-axis having coordinates in the GF(2).

b. By E_1 with a in the GF(2^2) gives a G_8 leaving the x-axis invariant. The system S includes all points on the x-axis.

c. By E_2 with b in the GF(2) extends by E with a and b in the GF(2) giving a G_8 leaving invariant a PG(2,2) which is also the system S.

d. By E_2 with b in the GF(2^2) gives a G_{32} leaving invariant no point not on l and no line but l. The system S includes all points not on l.

B. Extensions of G_4 of form T_4 with a_1, a_2 in GF(2^2). (5 of each).

a. By E_1 gives a G_{16} leaving invariant the x-axis.
The system S includes all points on the x-axis.

b. By E_2 extends by E giving G_{64} leaving invariant no point not on l and no line but l. The system S includes all points not on l.

C. Extension of G_{12} of form T_5: $x'=a_1x+a_2y$, $y'=b_2y$. (5 of each).

a. By E_1 gives a G_{48} leaving invariant the x-axis.
The system S includes all points on the x-axis.

b. By E_2 extends by E giving a G_{64} leaving invariant no point not on l and no line but l.
The system S includes all points on the x-axis.

2. Those of determinant not restricted. (5 of each).
   A. Extensions of G_{12} of form T_4.
      a. By E_1 gives a G_{48} leaving a point of lines invariant. The system S includes all points on the x-axis.
      b. By E_2 extends by E giving a G_{192} leaving invariant no point not on l and no line but l. The system S includes all points not on l.
   B. Extensions of G_{36} of form T_5.
      a. By E_1 gives a G_{144} leaving the x-axis invariant. The system S includes all points on the x-axis.
      b. By E_2 extends by E giving a G_{576} leaving invariant no point not on l and no line but l. The system S includes all points not on l.

IV. *Subgroups leaving invariant a pair of imaginary points on l.* (*6 of each*).
In each case the system S includes all points not on l and no point on l and no line other than l is invariant.
1. Subgroups leaving each point of the pair fixed.
   A. Of determinant unity.
      Extension of G_5 of form T_0 (subject to quadratic condition) by E_1 or E_2 extends by E giving a G_{80}.
   B. Of determinant not restricted.
      Extension of G_{15} of form T_0 (subject to quadratic condition) by E_1 or E_2 extends by E giving a G_{240}.
2. Subgroups leaving the points invariant as a pair.
   A. Of determinant unity.
      Extension of G_{10} of form T_0 (subject to cubic condition) by E_1 or E_2 extends by E giving a G_{160}.
   B. Of determinant not restricted.
      Extension of G_{30} of Form T_0 (subject to cubic condition) by E_1 or E_2 extends by E giving a G_{480}.

The extension of the total group G_{60} of the form T_0 and determinant unity by E_1 or E_2 extends by E giving a group G_{960} of all transformations of determinant unity in the G_{2880}. The extension of the group G_{180} of form T_0 by E_1 or E_2 extends by E giving the G_{2880} itself.

There remain to be determined the subgroups of G_{2880} which leave no point not on l invariant, contain translations, and are such that the self-conjugate subgroup of translations has no system of transitivity which is also a system of transitivity for all other transformations in the group. Since the group G_{16} of translations is transitive on all points not on l no such subgroup can contain the G_{16}.

Let G_n be such a subgroup of G_{2880} containing a G_2 as the largest self-conjugate subgroup of translations. Selecting $S: x'=x+1,\ y'=y$ as the translation in the G_2, and $T: x'=a_1x+b_1y+c_1,\ y'=a_2x+b_2y+c_2$ as the general transformation in the G_{2880} (cf. p. 31) we have $TST^{-1}: x'=x+a_1,\ y'=y+a_2$. Since $TST^{-1}\equiv S$ we have $a_1=1,\ a_2=0$ and every transformation in the G_n is of the form $T_1: x'=x+ay+b,\ y'=cy+d$. If $c=1$ T_1 is of period 2 if $cd=0$ and of period 4 if $ad=1$. If $c=i$ or i^2 and $b=acd$, T_1 is of period 3.. If $c=i$ or i^2 and b is not equal to acd, T is of period 6. Hence every transformation in G_n is of type II, III, IV or V. Also, all transformations of types III and V have for center the point $4\equiv(1,0,0)$ which is the center of S, all homologies have for axis some line other than l through 4 and for center some point other than 4 on l, and all transformations of type II have 4 as the point of intersection of the two invariant lines. Since no homology has l for axis, there are but two transformations (S and the identity) in G_n which leave l point-wise invariant. G_n is, therefore, at most (2, 1) isomorphic with some group in one dimension leaving a point on the line invariant and its possible orders are (cf. p. 32) 4, 8, 12 and 24. That G_n can be a cyclic G_4 follows from the fact that every system of transitivity of the G_2 contains but two points. G_n can not be a G_4 whose transformations are all of period 2, since if E be one of the elations in such a G_4, the G_2 and the G_4 have the same systems of transitivity on the axis of E. If G_n be of order 8 and contain transformations of period 2 only it must contain two elations, E_1 and E_2, having the same axis l_1, since such a G_8 must contain six elations and there are but four lines other than l through 4. We then have $E_1E_2 = E_3$ a third elation having l_1 for axis. Since E_1, E_2 and E_3 have the same center and axis they, together with the identity, form a group $G_4(P)$. The other three elations in the G_8 would be SE_1, SE_2 and SE_3; but since each of these elations has the same systems of transitivity on l_1 as S, G_n can not be such a G_8. Accordingly, if G_n be of order 8 it must contain a cyclic subgroup G_4 consisting of the powers of a transformation U of type III and since U must leave l invariant we have $U^2 \equiv S$. Hence U is of the form U: $x' = x + ay + b$, $y' = y + a^2$ where a is not zero. If the G_8 contain an elation it must be of the form E: $x' = x + a_1y + b_1$, $y' = y$ where a_1 is not zero. If $a = a_1$ the product EU: $x' = x + b + b_1 + 1$, $y' = y + a^2$ is a translation different from S. If a is not equal to a_1 the product EU: $x' = x + (a + a_1)y + b + b_1 + a_1a^2$, $y' = y + a^2$ is a transformation of type III whose square must be identical with S. If $(EU)^2$: $x' = x + a_1a^2 + 1$, $y' = y$ be identical with S we must have $a_1a^2 + 1 = 1$ or $a_1a^2 + 1 = 0$; but if $a_1a^2 + 1 = 1$, $a_1a^2 = 0$ which is impossible since neither a nor a_1 can be zero, and if $a_1a^2 + 1 = 0$, $a_1a^2 = 1$ which is impossible since a and a_1 are different marks of $GF(2^2)$. Hence if G_n be of order 8 it can contain no elations and must have 3 cyclic subgroups of order 4. Taking U_1: $x' = x + a_1y + b_1$, $y' = y + a_1^2$ as any other transformation of period 4 than U or U^3 in the G_8 we have UU_1: $x' = x + (a + a_1)y + aa_1^2 + b + b_1$, $y' = y + a^2 + a_1^2$ and U_1U: $x' = x + (a + a_1)y + a^2a_1 + b + b_1$, $y' = y + a^2 + a_1^2$. If $a = a_1$ the product UU_1 is a translation not in the G_8. Hence a must be different from a_1 and the group, if existent, is not Abelian and must be of the type* $U^4 = 1$, $U_1^4 = 1$, $U^2 = U_1^2$, $UU_1U^{-1} = U_1^{-1}$. Since these conditions are satisfied by any two transformations of the form U and U_1, where a is different from a_1 and neither a nor a_1 is zero, there are four such groups G_8 having the given G_2 as the largest self-conjugate subgroup of translations.

If G_n be of order 12 it must be simply isomorphic with the G_{12} consisting of all transformations leaving invariant a point on the line l (cf. III, p. 32). This is impossible since the product of the translation in G_n and any homology in G_n is of period 6. Hence G_n can not be of order 12. If G_n be of order 24 it must, by Theorem 5, contain exactly 8 transformations of determinant unity and 16 transformations of determinant i or i^2. The 8 transformations of determinant unity form a self-conjugate subgroup and hence the G_{24}, if existent, must be the direct product of this G_8 and a cyclic G_3 consisting of the powers of a homology H. It was shown above that a G_8 whose transformations are all of period 2 contains 3 elations E_1, E_1', E_1'', having the same axis l_1 and three other elations E_2, E_3, E_4, having for axes the three other lines l_2, l_3, l_4, respectively, through P. H

*Cf Burnside, l. c., p. 88.

can not have l_1 for axis since the G_2 and the G_{24} would then have the same systems of transitivity on l_1. Also H can not have l_2, l_3 or l_4 for axis, since if E_i $(i=2,3,4)$ be the corresponding elation, E_i and H are not commutative and HE_iH^{-1} would be an elation not in the G_8. Hence the self-conjugate G_8 in the G_{24} can not have all its transformations of period 2. Accordingly, the G_8 in the G_{24} must contain 3 cyclic subgroups of order 4. We may take this self-conjugate G_8 to be the one consisting of all transformations of the form $x'=x+ay+k$, $y'=y+a^2$ where a is any mark of the GF(2^2) and k is any mark of the GF(2). The three transformations in this G_8 of the form T: $x'=x+ay$, $y'=y+a^2$ where a is not zero are of period 4 and no two belong to the same cyclic G_4. The G_{24}, if existent, must contain a homology of the form H: $x'=x+by+bic$ $y'=iy+c$ and, therefore, every product TH: $x'=x+(ia+b)y+(a+ib)c$, $y'=iy+a^2+c$ where b and c are some two chosen marks of the GF(2^2). Since TH is of determinant i it is of type II or IV and, hence, $(\mathrm{TH})^3$: $x'=x+i(a^2b+ac+i)$, $y'=y$ must be identity or the translation S: $x'=x+1$, $y'=y$ for every value of a different from zero in the GF(2^2). But there exist no two marks b and c in the GF(2^2) which make $i(a^2b+ac+i)=m$ where m is in the GF(2) for every value of a different from zero in the GF(2^2). Hence, G_n can not be of order 24.

Subgroups having a $G_4(Q)$ as a self-conjugate subgroup. Let G_m be a sub group of G_{2880} having a $G_4(Q)$ as its largest self-conjugate subgroup of translations and such that the $G_4(Q)$ has no system of transitivity which is also a system of transitivity of the G_m. Selecting the $G_4(Q)$ as the group of all translations of the form S: $x'=x+a$, $y'=y+b$, where a and b are marks of the GF(2), and T is the general collineation (cf. p. 31) in the G_{2880} we have TST^{-1}: $x'=x+aa_1+bb_1$, $y'=y+aa_2+bb_2$. Hence if TST^{-1} belong to the $G_4(Q)$ $aa_1+bb_1=a'$ and $aa_2+bb_2=b'$ where a' and b' are in the GF(2). Since these equations must hold true for every a_1, a_2, b_1, b_2 in the G_m no matter what marks of the GF(2) a' and b' may be it follows that a_1, a_2, b_1, b_2 are in the GF(2) and every transformation in the G_m is of determinant unity. Consequently, G_m can contain no homology and is at most $(4, 1)$ isomorphic with some group on the line leaving invariant a pair of points. The possible orders of G_m are, therefore, (cf. p. 32) 12 and 24. The $G_4(Q)$ leaves each point on l $(x_3=o)$ invariant and permutes the other 16 points in four systems of transitivity each consisting of four points no three of which are collinear. These four quadrangles are $Q_1\equiv(0\ 16\ 20\ 18)$ $Q_2\equiv(1\ 14\ 8\ 5)$; $Q_2\equiv(2\ 13\ 6\ 15)$; $Q_4\equiv(9\ 10\ 12\ 11)$ (cf. Table of alignment, p. 3). Every transformation in G_m is of the form T: $x'=a_1x+b_1y+c_1$, $y'=a_2x+b_2y+c_2$ where a_1, a_2, b_1, b_2 are in the GF(2), and must, therefore, be of type V (elation, period 2), type I_3 (period 3) or type III (period 4). From the forms of T^2, T^3, T^4 it appears that every elation in G_m must be of the form E_1: $x'=y+k$, $y'=x+k$, or of the form E_2: $x'=x+y+c$, $y'=y$, or of the form E_3: $x'=x$, $y'=x+y+d$; every transformation of type I_3 must be of the form T_1: $x'=y+m$, $y'=x+y+n$, or of the form T_2: $x'=x+y+r$, $y'=x+s$; every transformation of type III must be of the form U_1: $x'=y+k$, $y'=x+l$ where l and k are not the same mark, or of the form U_2: $x'=x+y+c_1$, $y'=y+c_2$, or of the form U_3: $x'=x+d_1$, $y'=x+y+d_2$. If G_m be of order 12 it must be the direct product of a cyclic G_3(cyc. I_3) and the $G_4(Q)$. But each transformation of period 3 in G_m must leave one of the quadrangles Q_i $(i=1, 2, 3, 4)$ invariant and permute the other three in cyclic order. Hence some one of these quadrangles would be a system of transitivity of the G_{12} generated by any G_3(cyc. I_3) and the $G_4(Q)$ and G_m can not be or order 12. If G_m be of order 24 it must contain a transformation T of period 3 leaving invariant a quadrangle Q_a (a=1, 3, 3 or 4). Every transformation of the form T_1 or T_2 is seen to leave invariant the two points $7\equiv(i, 1, 0)$ and $19\equiv(1, i, 0)$ on l, and since every translation in the $G_4(Q)$ leaves every point on l invariant the eight products obtained by multiplying T and T^2 by the transformations in the $G_4(Q)$ are eight transformations of period 3 leaving Q_a invariant. Moreover, G_{24} can not contain any transformation T_1 of period 3 not included among these eight, for the products of T and the three other transformations leaving Q_a invariant and making the same permutation of points on l as T give the three translations in the $G_4(Q)$ and consequently one of the products T_1T or T_1T^2 would be a translation not in the $G_4(Q)$. Hence, the G_4 must contain, besides the $G_4(Q)$, 8 transformations of type I_3 leaving Q_a invariant and some transformation S_1 of period 2 or 4 transforming Q_a into one of the other quadrangles. Let S_1 be of period 2 or 4 interchanging the four quadrangles in any order $R_1=(Q_aQ_b)(Q_cQ_d)$ where a, b, c, d are the numbers 1, 2, 3, 4 in an arbitrary order. Since half of the transformations of period 3 in the G_{24} must make the transformation $R_2=(Q_a)(Q_bQ_cQ_d)$ on the four quadrangles Q_i (i=1, 2, 3, 4) the G_{24} would then contain a transformation of period 3 making on Q_i (i=1, 2, 3, 4) the transformation $R_2R_1=(Q_aQ_bQ_c)(Q_d)$ which has just been shown to be impossible. Hence the G_{24} can contain no transformation of period 2 or 4 interchanging the Q_i (i=1, 2, 3, 4) in pairs. Also, G_{24} can not contain a transformation of period 4 permuting the Q_i (i=1, 2, 3, 4) in cyclic order for its square would be a transformation of period 2 interchanging them by pairs. Hence, the G_{24} must contain a transformation S_1 of period 2 or 4 which transforms the Q_i (i=1, 2, 3, 4) in the order $R_3=(Q_aQ_b)(Q_c)(Q_d)$; but since $R_2R_3=(Q_aQ_bQ_cQ_d)$ the G_{24} would then contain transformations permuting the Q_i (i=1, 2, 3, 4) in cyclic order which has just been shown to be impossible. Hence G_{2880} contains no subgroup G_m having a $G_4(Q)$ as its largest self-conjugate subgroup of translations and such that the $G_4(Q)$ has no system of transitivity which is also a system of transitivity of the G_m.

Subgroups containing a self-conjugate $G_4(P)$. Let G_k be a subgroup of G_{2880} having a $G_4(P)$ as its largest self-conjugate subgroup of translations and such that the $G_4(P)$ has no system of transitivity which is also a system of transitivity of the G_k. Selecting the $G_4(P)$ as the group of all transformations of the form S: $x'=x+a$, $y'=y$, and T: $x'=a_1x+b_1y+c_1$, $y'=a_2x+b_2y+c_2$ as the general transformation in the G_{2880} we have TST^{-1}: $x'=x+a_1a$, $y'=y+aa_2$. Hence in order that TST^{-1} may belong to the $G_4(P)$ we must have $a_2=0$ and if we also have $a_1=1$ each translation in the $G_4(P)$ is self-conjugate. Hence every transformation in G_k is of the form T_1: $x'=a_1x+b_1y+c_1$, $y'=b_2y+c_2$. The $G_4(P)$ consists of all translations having l for axis and $4\equiv(1,0,0)$ for center. The $G_4(P)$ has four systems of transitivity, $l_1\equiv$(0 1 16 14), $l_2\equiv$(8 5 18 20), $l_3\equiv$(2 13 10 9), $l_4\equiv$(15 16 12 11) and in each system the four points are collinear. From the form of T_1^2 it appears that every elation in G_k is of the form E: $x'=x+b_1y+c_1$, $y'=y$ (where b_1 is not zero) and, consequently, has 4 for center and leaves each l_i (i=1, 2, 3, 4) invariant. Hence there is no G_k containing translations and elations only. From the form of T_1^4 it appears that every transformation of type III in G_k is of the form U: $x'=x+b_1y+c_1$, $y'=y+c_2$, where neither b_1 nor c_2 is zero. The 12 transformations for which $c_2=1$ make the interchange $(l_1l_2)(l_3l_4)$; the 12 for which $c_2=i$ make the interchange $(l_1l_3)(l_2l_4)$; and the 12 for which $c_2=i^2$ make the interchange $(l_1l_4)(l_3l_2)$. Since the G_k can not leave any l_i (i=1, 2, 3, 4) invariant it must either be transitive on the l_i or interchange them by pairs. Suppose G_k to make the interchange $(l_1l_2)(l_3l_4)$. Such a G_k can not contain a homology having l for axis, for the homology would permute three of the lines l_i in cylic order. Hence, the G_k would be at most (4, 1) isomorphic with some group on the line leaving a point invariant and its possible orders would be (cf. p. 32) 8, 12, 16, 24, 48. Since G_k is transitive on 8 points its order must be divisible by 8 and can not be 12. If G_k be of order 8 and make the interchange $(l_1l_2)(l_3l_4)$ it must contain a transformation of the form U_1: $x'=x+b_1y+c_1$, $y'=y+i$. The products of a U_1 and the transformations in the $G_4(P)$ are four transformations of the form U_1 where b_1 is fixed and c_1 is any mark of the $GF(2^2)$. These 4 transformations of type III and the transformations in the $G_4(P)$ form a group G_8 which is a G_k. Hence there are three such subgroups G_8 (one for each value of b_1) interchanging the l_i (i=1, 2, 3, 4) in the order $(l_1l_2)(l_3l_4)$ and similarly 3 subgroups G_8 for each of the orders $(l_1l_3)(l_2l_4)$ and $(l_1l_4)(l_2l_3)$. No G_k can contain more than 8 transformations of the form U_1, since the product of U_1: $x'=x+y+c_1$, $y'=y+$ U_1': $x'=x+iy+c_1$, $y'=y+i$, and U_1'': $x'=x+i^2y+c_1$, $y'=y+i$, is $U_1\,U_1'\,U_1''$: $x'=x+c_1+i$, $y'=y+i$, which is a translation not in the $G_4(P)$. The product of any two transformations of the form U_1 and U_1' is $U_1U_1'=E_1$: $x'=x+i^2y+i$, $y'=y$ which is an elation having 4 for center. The products of E_1 and the four transformations in the $G_4(P)$ give all elations of the form E_1': $x'=x+i^2y+c$, $y'=y$. The product of any two elations of the form E_1' is a translation in the $G_4(P)$. Also all products of transformations of the form E_1' with transformations of the form U_1 or U_1' are of the form U_1' or U_1 respectively. Hence, the 12 transformations of the forms U_1, U_1' and E_1' together with the $G_4(P)$ form a Group G_{16} which is a G_k. Obviously there may be two other such groups G_{16} making the interchange $(l_1\ l_2)(l_3\ l_4)$ and similarly 3 groups G_{16} making the interchange $(l_1\ l_3)(l_2\ l_4)$ and 3 groups G_{16} making the interchange $(l_1\ l_4)(l_2\ l_3)$. Also, since the product of an elation of the form E: $x'=x+by+c$, $y'=y$ and a transformation of type III of the form U: $x'=x+by+c_1$, $y'=y+c_2$ is EU: $x'=x+c_1+bc_2+c$, $y'=y+c_2$, which is a translation not in the $G_4(P)$, there is no other type of group of order 16 which can be a G_k interchanging the l_i by pairs.

A G_k of order 24 or 48 which makes the interchange $(l_1l_2)(l_3l_4)$ must contain transformations of period 3. Every such transformation would leave each l_i invariant and hence would be a homology H having 4 for center. Such a G_k must also contain some transformation T making the interchange $(l_1\ l_2)(l_3\ l_4)$. But T can not be of type II, for in that case T^3 would be a translation not in the $G_4(P)$, and T can not be of type III, for in that case HT would be such a transformation of type II. Hence there is no G_k of order 24 or 48 interchanging the l_i by pairs..

No G_k can contain a homology H having l for axis; for, if P_1 be the center of H, G_k must contain some transformation T transforming P_1 to some point P_1' which is not collinear with P_1 and 4. Since H can not leave P_1' invariant HT and TH transform P_1 to different positions and, hence, H and T are not commutative $THT^{-1}=H_1$ is, therefore, a homology having P_1 for center and one of the products HH_1 or H^2H_1 is a translation not in the $G_4(P)$ since it would have for center the point where the line P_1P_1' cut l. Accordingly G_k is at most (4, 1) isomorphic with some group on the line leaving a point invariant and its possible orders are 8, 12, 16, 24 or 48. If G_k be transitive on the l_i it is transitive on all points not on l and its order must, therefore, be divisible by 16. Consequently such a transitive G_k must be of order 16 or 48. If such a G_k be of order 16 it must contain two transformations of type III, U_1: $x'=x+a_1y+b_1$, $y'=y+c_1$, and U_2: $x'=x+a_2y+b_2$, $y'=y+c_2$ where c_1 and c_2 are neither one zero and are distinct from each other. The product U_1U_2 is of the form U_3: $x'=x+a_3y+b_3$, $y'=y+c_3$ where a_3 is different from a_1 and a_2 and c_3 is different from c_1 and c_2. Also a_1 must be distinct from a_2, for otherwise U_3 is a translation not in the $G_4(P)$. All products U_2U_3 are of the form U_1 and all products U_1U_3 are of the form U_2. Also, the square of each $U_i (i=1, 2, 3)$ is a translation in the $G_4(P)$. Hence 12 transformations, 4 each of the forms U_1, U_2, U_3, such that a_1, a_2, a_3 are no two the same mark and c_1, c_2, c_3, are no two the same mark, form, together with the $G_4(P)$ a group and the only type of group G_{16} which is a transitive G_k of order 16. Since a_1 and c_1 may each be chosen in 3 different ways and a_2 and c_2 may each then be chosen in 2 different ways there are 36 such groups G_{16} having the given $G_4(P)$ as its largest self-conjugate subgroup of translations.

If G_k be a transitive group of order 48 it must contain a transformation T of period 3. If T be a homology it must either have some line through 4 for axis or have 4 for center. That T can not be a homology, having l for axis has been shown above. That T can not be a homology having any other line through 4 for axis follows from the fact that if S be one of the translations in the $G_4(P)$ TS is of type II and $(TS)^3$ is a translation not in the $G_4(P)$. If T be a homology having 4 for center it leaves each $l_i (i=1, 2, 3, 4)$ invariant. Since the G_{48} must contain a subgroup G_{16} which can have no transformation other than identity in common with the cyclic G_3 generated by T, the group $\{G_{16}, G_3\}$ is the G_{48} and leaves each $l_i (i=1, 2, 3, 4)$ invariant unless the G_{16} contain a transformation U of type III having 4 for center and having for axis a line through P_3 some point on l different from P_1 and P_2. Hence the G_{48} would contain at least 24 distinct homologies and since the products of these by U give 24 distinct transformations of type II the G_{48} would contain more than 48 transformations. Hence a transitive G_k of order 48 can not contain a homology.

Since T can not be a homology it must be of type I_3 having for vertices of its invariant triangle a point A not on l and two points, 4 and some other point P_1, other than 4 on l. The cyclic G_3 generated by T together with the $G_4(P)$ generates a G_{12} consisting of all transformations of determinant unity leaving invariant the points 4 and P_1 and the line l_a joining A to 4. But the G_{48} must contain a subgroup G_{16} having no transformation other than identity in common with the cyclic G_3 generated by T. Hence the G_{16} and the cyclic G_3 would generate a G_{48} leaving l_a invariant unless the G_{16} contain a transformation U of type III interchanging l_a with some other line l_b through 4. If the other two lines through 4 be designated as l_c and l_d, U makes the transformation $U=(l_al_b)(l_cl_d)$ on these lines and T makes the transformation $T=(l_a)(l_bl_cl_d)$. Hence $TU=(l_al_cl_b)(l_d)$ is of type I_3 leaving l_d invariant and $UT=(l_al_bl_d)(l_c)$ is of type I_3 leaving l_c invariant. Also, the product of TU and UT is a transformation of type I_3 leaving the line l_a invariant. Each of these transformations of type I_3 generates a cyclic G_3, which, taken with the $G_4(P)$ generates a G_{12} containing 8 transformations of type I_3. These 32 transformations of type I_3 are such that each point not on l is an invariant point of two of them and each point on l (other than 4) is an invariant point of 8 of them. Hence the self-conjugate G_{16} can not contain an elation E; for, if T′ be a transformation of type I_3 having an invariant point on the axis of E, ET′ would be a transformation of type I_3 not among the 32 above named. The G_{16} must, therefore, consist of 12 transformations of type III and the $G_4(P)$. Taking the G_{16} as all transformations of the form U: $x'=x+ay+b$, $y'=y+a$ and the cylic G_3 generated by a transformation of type I_3 as all transformations of the form T: $x'=mx$, $y'=n$, where $mn=1$ and neither m nor n is unity, the products UT: $x'=mx+any+b$, $y'=ny+a$ and TU: $x'=mx+amy+mb$, $y'=ny+na$ are 32 transformations of type I_3 as above described. Also, it is readily verified that the product of any two transformations of the form U, UT and TU is of the form U, UT or TU. Hence they form a group G_{48} which is a transitive G_k.

Subgroups containing a self-conjugate G_8. There remains to be determined every subgroup G_t of G_{2880} which leaves fixed no point not on l, has a G_8 as its largest self-conjugate subgroup of translations and has no system of transitivity which is also a system of transitivity of the G_8. A G_8 of translations consists of a $G_4(P)$ (all translations of which have for center the same point P on l) and 4 other translations each having a different center from any other in the G_8. The G_8 leaves invariant besides l and P a pair of lines through P. Hence the G_8 has two systems of transitivity of 8 points each and a G_t must be transitive on the 16 points not on l. No G_t can contain a homology H having l for axis; for, if T be a translation in the G_8 having a point P′ different from P for center H and T are not commutative (since HT and TH transform the center of H to different positions) and HTH^{-1} is a translation having P′ for center and not in the G_8. Hence, a G_t contains no transformation leaving l pointwise invariant except the translations in the G_8 and is at most (8, 1) isomorphic with some group leaving invariant a point on the line. Its possible orders are, therefore, (cf. p. 32) 16, 32, 48 and 96. Taking P as the point $4=(1,0,0)$ the $G_4(P)$ in the G_8 becomes the four translations of the form S: $x'=x+a, y'=y$ and the four other translations in the G_8 may be taken as the four translations of the form S_1: $x'=x+a$, $y'=y+1$. Since a G_t can contain no translation not in the G_8 a translation of the form S must be transformed into a translation of the form S, and a transformation of the form S_1 must be transformed into a translation of the form S_1 by every transformation in G_t. It has already appeared above that the first of these conditions requires that every transformation in a G_t shall be of the form T_1: $x'=a_1x+b_1y+c_1$, $y'=b_2y+c_2$. Transforming S_1 through T_1 gives $TS_1T_1^{-1}$: $x'=x+aa_1$, $y'=y+b_2$ and hence $a_1=1$, $b_2=1$ and every transformation in G_t is of the form T_2: $x'=x+b_1y+c_1$, $y'=y+c_2$. If $b_1=0$ or if $c_2=0$ T_2 is of period 2. Hence, every elation in a G_t is of the form E: $x'=x+b_1y+c_1, y'=y$, where b_1 is not zero. If neither b_1 nor c_2 is zero T_1 is of period 4. Accordingly a G_t can contain only transformations of period 2 or 4 and must be of order 16 or 32.

The two systems of transitivity of the G_8 are left invariant by every elation of the form E and hence a G_t must contain a transformation U of type III. Under the G_8 of translations the pair of lines $y=0$, $y=1$ is one system of transitivity and the pair of lines $y=i$, $y=i^2$ (equations in non-homogeneous coordinates) is the other system. Hence U must be of the form U_1: $x'=x+ay+b$, $y'=y+i$ or of the form U_2: $x'=x+ay+b$, $y'=y+i^2$. But the product of a translation of the form S_1 and a transformation of the form U_1 or U_2 is of the form U_2 or U_1, respectively, and hence every G_t must contain transformations of both forms. A G_t of order 16 must, therefore, contain, besides the G_8 of translations, 4 transformations of the form U_1 and 4 of the form U_2 where a has the same value for all of these transformations of type III. Since every product U_1U_2 and U_2U_1 is a translation in the G_8 these 16 transformations form a Group G_{16} which is a G_t. Since there are 3 choices for a there are 3 such groups G_{16} having the given G_8 as the largest subgroup of translations. Also, it appears from the above that every G_t must contain at least one such G_{16} as a subgroup. This G_{16} will be taken as the subgroup consisting of the 8 translations in the G_8, 4 transformations of the form U_1': $x'=x+y+c$, $y'=y+i$ and the four transformations of the form U_2': $x'=x+y+c$, $y'=y+i^2$. For convenience this G_{16} will be referred to as the group G'.

A G_t of order 32 must contain some transformation T of period 2 or 4 not in the subgroup of order 16 taken as G'. If T be of period 2 it must be of the form E: $x'=x+ay+b$, $y'=y$. The products $U_1'E$: $x'=x+(a+1)y+b+c$, $y'=y+i$, and $U_2'E$: $x'=x+(a+1)y+b+c$, $y'=y+i^2$ are translations not in the G_8 of translations if $a=1$. If $a=i$, the G_{32} must contain 4 transformations of type III of the form U_1'': $x'=x+i^2y+m$, $y'=y+i$, and 4 of the form U_2'': $x'=x+i^2y+m$, $y'=y+i^2$. Also, the products of E and the translations of the form S_1 introduce 4 transformations of the form U_3: $x'=x+iy+b$, $y'=y+1$. Thus are determined, besides the 8 translations in the G_8, 4 transformations of each of the forms E, U_1', U_1'' and U_3. These are all of the form U: $x'=x+ay+b$, $y'=y+c$ where $c=i$ or i^2 if $a=1$ or i^2 and $c=0$ or 1 if $a=0$ or i. Taking U_1: $x'=x+a_1y+b_1$, $y'=y+c_1$ as a second transformation of the form U the product is UU_1: $x'=x+(a_1+a)y+ac_1+b_1+b$, $y'=y+c+c_1$. If $a=0$ or $a_1=0$ or if $a=a_1$ it may be verified by inspection that this product is one of the given forms. For the other possibilities the following table gives the results:

a	a_1	c	c_1	*Resulting forms.*
1	i	i^2 or i	0 or 1	U_1'' or U_2''
1	i^2	i^2 or i	i^2 or i	E or U_3
i	1	0 or 1	i^2 or i	U_1'' or U_2''
i	i^2	0 or 1	i or i^2	U_1' or U_2'
i^2	1	i or i^2	i or i^2	E or U_3
i^2	i	i or i^2	0 or 1	U_1' or U_2'

Hence these 32 transformations form a group G_{32} which is a G_t.

If a in E had been chosen as i^2, a G_{32} of the same type would have been determined. Hence there are two groups G_{32} having the G' as a subgroup.

SUMMARY.

1. In the finite projective plane $PG(2,2^n)$ the diagonal points of a complete quadrangle are collinear.

2. If an outside point of a conic be defined as any point of intersection of tangents to the conic, every conic in the $PG(2,2^n)$ has but one outside point and all tangents to the conic concur at that point. Through every point other than the outside point there passes one and but one tangent to the conic and every line through the outside point of a conic is a tangent to the conic.

3. In the $PG(2,2^n)$ six and but six points can be chosen such that no three of the set are collinear.

4. All of the types of projective collineations of the ordinary projective plane are present in the $PG(2,2^n)$ and the number of such collineations in the $PG(2,2^2)$ is 60480.

5. Every subgroup of the group G_{60480} of all projective collineations in the $PG(2,2^2)$, except a self-conjugate G_{20160} leaves invariant a real figure [real within the $PG(2,2^2)$] or an imaginary triangle.

6. There are 8 kinds of groups leaving invariant an imaginary triangle and their list is given in Theorem 11.

7. All configurations in the $PG(2,2^2)$ and the groups characterizing them are determined. These groups include the finite groups of the ordinary projective plane. Consequently, the simple G_{360}, the Hessian G_{216} and the simple G_{168} are all subgroups of the G_{60480} and within the $PG(2,2^2)$ the geometric invariant of each is a real configuration.

8 The subgroups of the G_{2880} which leaves a line invariant are chiefly (1,1) or (3,1) isomorphic with groups on the line, but certain groups of higher isomorphism are present and are determined.

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